Homework 6: Curvatures for Surfaces

(Total 20 pts + bonus 5 pts; Due Oct. 28 12pm)

Question 1. (10 pts) *Calculate H, K, κ_1, κ_2, t_1, t_2 at the point $(1,1,1)$ for the surface $z = x\,y$.*

Solution. We take the natural surface patch $\sigma(u,v) = (u, v, u\,v)$. Then we have

$$\sigma_u = (1,0,v), \quad \sigma_v = (0,1,u), \quad \sigma_{uu} = \sigma_{vv} = (0,0,0), \quad \sigma_{uv} = (0,0,1), \tag{1}$$

$$N = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}. \tag{2}$$

Thus at $(1,1,1)$ which corresponds to $u = v = 1$, we have

$$\sigma_u = (1,0,1), \sigma_v = (0,1,1), \sigma_{uu} = \sigma_{vv} = (0,0,0), \sigma_{uv} = (0,0,1), N = \frac{(-1,-1,1)}{\sqrt{3}}. \tag{3}$$

Consequently

$$\mathbb{E} = 2, \mathbb{F} = 1, \mathbb{G} = 2, \mathbb{L} = \mathbb{N} = 0, \mathbb{M} = \frac{1}{\sqrt{3}}. \tag{4}$$

Solving the equation

$$\det\begin{pmatrix} \mathbb{L} - \kappa_i\,\mathbb{E} & \mathbb{M} - \kappa_i\,\mathbb{F} \\ \mathbb{M} - \kappa_i\,\mathbb{F} & \mathbb{N} - \kappa_i\,\mathbb{G} \end{pmatrix} = 0 \tag{5}$$

we obtain $\kappa_1 = \frac{1}{3\sqrt{3}}, \kappa_2 = -\frac{1}{\sqrt{3}}$. Now we have

$$\begin{pmatrix} \mathbb{L} - \kappa_1\,\mathbb{E} & \mathbb{M} - \kappa_1\,\mathbb{F} \\ \mathbb{M} - \kappa_1\,\mathbb{F} & \mathbb{N} - \kappa_1\,\mathbb{G} \end{pmatrix} = \frac{2}{3\sqrt{3}}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6}$$

Thus

$$\frac{2}{3\sqrt{3}}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Longrightarrow t_1 = \frac{(1,1,2)}{\sqrt{6}}. \tag{7}$$

Similarly we have

$$t_2 = \frac{(1,-1,0)}{\sqrt{2}}. \tag{8}$$

Finally we easily obtain

$$H = \frac{\kappa_1 + \kappa_2}{2} = -\frac{1}{3\sqrt{3}}, \qquad K = \kappa_1\,\kappa_2 = -\frac{1}{9}. \tag{9}$$

Question 2. (5 pts) *Let $\sigma(u,v)$ be a surface patch. Assume that $\mathbb{E} = \mathbb{G}, \mathbb{F} = 0$.*

a) (3 pts) *Prove that $\sigma_{uu} + \sigma_{vv}$ is perpendicular to σ_u and σ_v.*

b) (2 pts) *Prove that if $\sigma_{uu} + \sigma_{vv} = 0$ then $H = 0$.*

Proof.

a) We have

$$\begin{aligned}(\sigma_{uu}+\sigma_{vv})\cdot\sigma_u &= \sigma_{uu}\cdot\sigma_u+\sigma_{vv}\cdot\sigma_u\\ &= \left(\frac{\mathbb{E}}{2}\right)_u+(\sigma_u\cdot\sigma_v)_v-\sigma_{uv}\cdot\sigma_v\\ &= \left(\frac{\mathbb{E}}{2}\right)_u+\mathbb{F}_v-\left(\frac{\mathbb{G}}{2}\right)_u\\ &= 0.\end{aligned}\tag{10}$$

Similarly we have $(\sigma_{uu}+\sigma_{vv})\cdot\sigma_v=0$.

b) As $\sigma_{uu}+\sigma_{vv}=0$ we have $\mathbb{L}+\mathbb{N}=0$. Then using $\mathbb{E}=\mathbb{G}, \mathbb{F}=0$ we have

$$H=\frac{\mathbb{E}\,\mathbb{N}+\mathbb{L}\,\mathbb{G}-2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)}=\frac{\mathbb{E}\,(\mathbb{L}+\mathbb{N})}{2\,\mathbb{E}^2}=0.\tag{11}$$

Thus ends the proof. □

QUESTION 3. (5 PTS) *Let a surface S be such that $H=0, K\neq 0$. Prove that its Gauss map $\mathcal{G}$ is conformal. That is if σ is a surface patch for S and we take $\tilde{\sigma}:=\mathcal{G}\circ\sigma$ as the surface patch for $\mathbb{S}^2$, then there is a scalar function λ such that $\tilde{\mathbb{E}}=\lambda\,\mathbb{E}, \tilde{\mathbb{F}}=\lambda\,\mathbb{F}, \tilde{\mathbb{G}}=\lambda\,\mathbb{G}$.*

(Hint: Write $-N_u=a_{11}\,\sigma_u+a_{12}\,\sigma_v, -N_v=a_{21}\,\sigma_u+a_{22}\,\sigma_v$ and recall the relations between a_{ij} and $\mathbb{E},\mathbb{F},\mathbb{G},\mathbb{L},\mathbb{M},\mathbb{N},H,K$. It also helps to write things in matrix form.)

Proof. The surface patch $\tilde{\sigma}$ is in fact just $N(u,v)=\mathcal{G}(\sigma(u,v))$. We write

$$-N_u=a_{11}\,\sigma_u+a_{12}\,\sigma_v,\qquad -N_v=a_{21}\,\sigma_u+a_{22}\,\sigma_v.\tag{12}$$

Then we have

$$\begin{pmatrix}\tilde{\mathbb{E}} & \tilde{\mathbb{F}}\\ \tilde{\mathbb{F}} & \tilde{\mathbb{G}}\end{pmatrix}=\begin{pmatrix}N_u\cdot N_u & N_u\cdot N_v\\ N_u\cdot N_v & N_v\cdot N_v\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}.\tag{13}$$

Now recalling $\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}=\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}$, we see that

$$\begin{pmatrix}\tilde{\mathbb{E}} & \tilde{\mathbb{F}}\\ \tilde{\mathbb{F}} & \tilde{\mathbb{G}}\end{pmatrix}=\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}=\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}^2.\tag{14}$$

Next since $0=H=a_{11}+a_{22}$, we have

$$\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}^2=\begin{pmatrix}a_{11}^2+a_{21}\,a_{12} & a_{11}\,a_{21}+a_{22}\,a_{21}\\ a_{11}\,a_{12}+a_{22}\,a_{12} & a_{22}^2+a_{12}\,a_{21}\end{pmatrix}=(a_{12}\,a_{21}-a_{11}\,a_{22})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\tag{15}$$

Consequently there holds

$$\begin{pmatrix}\tilde{\mathbb{E}} & \tilde{\mathbb{F}}\\ \tilde{\mathbb{F}} & \tilde{\mathbb{G}}\end{pmatrix}=-\left[\det\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\right]\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}=-K\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix},\tag{16}$$

as desired. □

The following are more abstract or technical questions. They carry bonus points.

QUESTION 4. (**BONUS, 5 PTS**) *Let S be a surface and $p_0 \in S$. Assume that there is a surface patch σ covering $p_0 = \sigma(u_0, v_0)$ such that at p_0 there holds $\mathbb{E} = \mathbb{G} = 1$, $\mathbb{F} = 0$ and $\mathbb{E}_u = \mathbb{E}_v = \mathbb{F}_u = \mathbb{F}_v = \mathbb{G}_u = \mathbb{G}_v = 0$.*

a) (3 PTS) *Prove that the Gaussian curvature at p is*
$$K = \frac{\partial^2 \mathbb{F}}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 \mathbb{G}}{\partial u^2} - \frac{1}{2}\frac{\partial^2 \mathbb{E}}{\partial v^2}. \tag{17}$$

b) (1 PT) *Let $\tilde{S}$ be another surface such that there is a local isometry $f: S \mapsto \tilde{S}$. Prove that the Gaussian curvature at $f(p)$ is $\tilde{K} = K$.*

c) (1 PT) *Prove or disprove: $\tilde{H} = H$.*

Proof.

a) We calculate
$$\begin{aligned} \frac{\partial^2 \mathbb{F}}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 \mathbb{G}}{\partial u^2} - \frac{1}{2}\frac{\partial^2 \mathbb{E}}{\partial v^2} &= (\sigma_u \cdot \sigma_v)_{uv} - \left(\frac{\sigma_v^2}{2}\right)_{uu} - \left(\frac{\sigma_u^2}{2}\right)_{vv} \\ &= (\sigma_{uuv} \cdot \sigma_v + \sigma_{uu} \cdot \sigma_{vv} + \sigma_{uv} \cdot \sigma_{uv} + \sigma_u \cdot \sigma_{uvv}) \\ &\quad -(\sigma_{uv} \cdot \sigma_{uv} + \sigma_v \cdot \sigma_{uuv}) - (\sigma_{uv} \cdot \sigma_{uv} + \sigma_u \cdot \sigma_{uvv}) \\ &= \sigma_{uu} \cdot \sigma_{vv} - \sigma_{uv} \cdot \sigma_{uv}. \end{aligned} \tag{18}$$

Now at p_0 write
$$\sigma_{uu} = a\,\sigma_u + b\,\sigma_v + \mathbb{L}\,N. \tag{19}$$

As $\mathbb{E} = \mathbb{G} = 1, \mathbb{F} = 0$ we see that σ_u, σ_v, N form an orthonormal basis.

Since $\mathbb{E}_u = 0$ we have $\sigma_{uu} \cdot \sigma_u = 0$ (at p_0 only!) so $a = 0$. Since $\mathbb{F}_u = \mathbb{G}_v = 0$ we have $\sigma_{uu} \cdot \sigma_v = 0$ so $b = 0$. Thus we have $\sigma_{uu} = \mathbb{L}\,N$.

Similarly we can prove $\sigma_{vv} = \mathbb{N}\,N$ and $\sigma_{uv} = \mathbb{M}\,N$. Consequently
$$\sigma_{uu} \cdot \sigma_{vv} - \sigma_{uv} \cdot \sigma_{uv} = \mathbb{L}\,\mathbb{N} - \mathbb{M}^2 = K \tag{20}$$
and the proof ends.

b) Let σ be a surface patch for S and let $\tilde{\sigma} := f \circ \sigma$. Then $\tilde{\sigma}$ is a surface patch for $\tilde{S}$ and we have, thanks to f being a local isometry, $\tilde{\mathbb{E}} = \mathbb{E}, \tilde{\mathbb{F}} = \mathbb{F}, \tilde{\mathbb{G}} = \mathbb{G}$. Now $\tilde{K} = K$ immediately follows from (17).

c) This is not correct. For example consider plane and cylinder. Note that when S is part of a plane, we can take an orthonormal basis and obtain $\mathbb{E} = \mathbb{G} = 1, \mathbb{F} = 0$. Thus the assumptions are satisfied.

□