Homework 5: The Second Fundamental Form

(Total 20 pts + bonus 5 pts; Due Oct. 14 12pm)

Question 1. (5 pts) *Calculate the second fundamental form of the surface*

$$\sigma(u,v) = (u, v, u\,v). \tag{1}$$

Solution. We have

$$\sigma_u = (1, 0, v), \quad \sigma_v = (0, 1, u), \tag{2}$$

$$\sigma_{uu} = (0,0,0), \quad \sigma_{uv} = (0,0,1), \quad \sigma_{vv} = (0,0,0). \tag{3}$$

Furthermore we have

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-v, -u, 1)}{\sqrt{1+u^2+v^2}}. \tag{4}$$

Therefore

$$\mathbb{L} = \mathbb{N} = 0, \qquad \mathbb{M} = \frac{1}{\sqrt{1+u^2+v^2}} \tag{5}$$

and the second fundamental form reads $\frac{\mathrm{d}u\,\mathrm{d}v}{\sqrt{1+u^2+v^2}}$.

Question 2. (5 pts) *Let $f: S_1 \mapsto S_2$ be a local isometry. Let σ_1 be a surface patch for S_1 and let $\sigma_2 = f \circ \sigma_1$. Prove or disprove: There hold $\mathbb{L}_2 = \mathbb{L}_1, \mathbb{M}_2 = \mathbb{M}_1, \mathbb{N}_2 = \mathbb{N}_1$.*

Solution. This is clearly false. Consider $\sigma_1(u,v) = (u, v, 0)$ and $f(x, y, 0) = (\cos x, \sin x, y)$. Then

$$\mathbb{L}_1 = \mathbb{M}_1 = \mathbb{N}_1 = 0 \tag{6}$$

but $\mathbb{L}_2 = -1$.

Question 3. (10 pts) *Let $x(s) = \sigma(u(s), v(s))$ be an arc length parametrized curve on a surface patch σ. Assume that at every s the curvature of $x(s)$, $\kappa(s) = |\kappa_n(s)|$, the absolute value of the normal curvature of the surface.*

a) (5 pts) *Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,u' + \mathbb{F}\,v') = \frac{1}{2}(\mathbb{E}_u\,(u')^2 + 2\,\mathbb{F}_u\,u'\,v' + \mathbb{G}_u\,(v')^2), \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\,u' + \mathbb{G}\,v') = \frac{1}{2}(\mathbb{E}_v\,(u')^2 + 2\,\mathbb{F}_v\,u'\,v' + \mathbb{G}_v\,(v')^2). \tag{8}$$

b) (5 pts) *Let S be the cylinder $\sigma(u,v) = (\cos u, \sin u, v)$. Find all the curves $x(s)$ on S satisfying $\kappa(s) = |\kappa_n(s)|$ for every s. Note that to solve b) you don't need to know how to prove a).*

Proof.

a) We recall that

$$\kappa(s) = |u''(s)\,\sigma_u + v''(s)\,\sigma_v + u'(s)^2\,\sigma_{uu} + 2\,u'(s)\,v'(s)\,\sigma_{uv} + v'(s)^2\,\sigma_{vv}| \tag{9}$$

and

$$\kappa_n(s) = (u'(s)^2\, \sigma_{uu} + 2\, u'(s)\, v'(s)\, \sigma_{uv} + v'(s)^2\, \sigma_{vv}) \cdot N. \tag{10}$$

Thus if $\kappa(s) = |\kappa_n(s)|$, necessarily

$$(u''(s)\, \sigma_u + v''(s)\, \sigma_v + u'(s)^2\, \sigma_{uu} + 2\, u'(s)\, v'(s)\, \sigma_{uv} + v'(s)^2\, \sigma_{vv}) \parallel N \tag{11}$$

or equivalently

$$(u''(s)\, \sigma_u + v''(s)\, \sigma_v + u'(s)^2\, \sigma_{uu} + 2\, u'(s)\, v'(s)\, \sigma_{uv} + v'(s)^2\, \sigma_{vv}) \cdot \sigma_u = 0 \tag{12}$$

and

$$(u''(s)\, \sigma_u + v''(s)\, \sigma_v + u'(s)^2\, \sigma_{uu} + 2\, u'(s)\, v'(s)\, \sigma_{uv} + v'(s)^2\, \sigma_{vv}) \cdot \sigma_v = 0. \tag{13}$$

Now we prove that (12) is equivalent to (7). The proof for (13) $\Longleftrightarrow$ (8) is almost identical and omitted.

Notice that

$$\sigma_u \cdot \sigma_u = \mathbb{E}, \qquad \sigma_v \cdot \sigma_u = \mathbb{F}, \tag{14}$$

$$\sigma_{uu} \cdot \sigma_u = \left(\frac{\sigma_u \cdot \sigma_u}{2} \right)_u = \frac{1}{2}\, \mathbb{E}_u, \quad \sigma_{uv} \cdot \sigma_u = \left(\frac{\sigma_u \cdot \sigma_u}{2} \right)_v = \frac{1}{2}\, \mathbb{E}_v, \tag{15}$$

and finally

$$\sigma_{vv} \cdot \sigma_u = (\sigma_v \cdot \sigma_u)_v - \sigma_v \cdot \sigma_{vu} = \mathbb{F}_v - \frac{1}{2}\, \mathbb{G}_u. \tag{16}$$

Substituting these into (12) we obtain

$$\mathbb{E}\, u'' + \mathbb{F}\, v'' + \frac{1}{2}\, \mathbb{E}_u\, (u')^2 + \mathbb{E}_v\, u'\, v' + \mathbb{F}_v\, (v')^2 - \frac{1}{2}\, \mathbb{G}_u\, (v')^2 = 0. \tag{17}$$

On the other hand, expanding the left hand side of (7) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\, u' + \mathbb{F}\, v') &= \frac{\mathrm{d}\mathbb{E}(u(t), v(t))}{\mathrm{d}t}\, u' + \frac{\mathrm{d}\mathbb{F}(u(t), v(t))}{\mathrm{d}t}\, v' + \mathbb{E}\, u'' + \mathbb{F}\, v'' \\ &= \mathbb{E}_u\, (u')^2 + \mathbb{E}_v\, u'\, v' + \mathbb{F}_u\, u'\, v' + \mathbb{F}_v\, (v')^2 + \mathbb{E}\, u'' + \mathbb{F}\, v''. \end{aligned} \tag{18}$$

The conclusion now trivially follows.

b) For the cylinder we have $\sigma_u = (-\sin\, u,\, \cos\, u,\, 0)$ and $\sigma_v = (0,\, 0,\, 1)$. Consequently $\mathbb{E} = \mathbb{G} = 1, \mathbb{F} = 0$. Thus (7), (8) becomes

$$u''(s) = 0, \qquad v''(s) = 0. \tag{19}$$

Therefore the curves satisfying $\kappa(s) = |\kappa_n(s)|$ are σ-images of straight lines. $\square$

The following are more abstract or technical questions. They carry bonus points.

Question 4. (**Bonus, 5 pts**) *A normal section of a surface S is the intersection between S and a plane Π that is perpendicular to the tangent plane of the surface at every point of this intersection curve. Assume that at every $p \in S$, for every $w \in T_p S$, the plane spanned by the surface normal N and w intersects S along a normal section of S. Prove that S is part of a sphere.*

Proof. Fix an arbitrary $p \in S$. Let $w := w_1 \sigma_u + w_2 \sigma_v \in T_p S$ also be arbitrary. Let Π be the plane passing p spanned by N and w. Let C be the curve $\Pi \cap S$.

As at every $p' \in C$ we have $N_{p'} \in \Pi$, the directional derivative $\frac{\partial N}{\partial w}$ must be a linear combination of N and w. But N is a unit vector so $\frac{\partial N}{\partial w} \perp N$. Consequently $\frac{\partial N}{\partial w} \parallel w$. In other words, there holds

$$(w_1 N_u + w_2 N_v) \times (w_1 \sigma_u + w_2 \sigma_v). \tag{20}$$

As (20) holds for every $w_1, w_2 \in \mathbb{R}$, in particular it holds for $(w_1, w_2) = (1, 0), (0, 1), (1, 1)$. Substituting the first into (20) we obtain $N_u \parallel \sigma_u \Longrightarrow N_u = \lambda \sigma_u$ for some λ; substituting the second into (20) we have $N_v = \mu \sigma_v$; then the third gives $\lambda = \mu$.

Consequently we have $\mathbb{L}\, \mathrm{d}u^2 + 2\, \mathbb{M}\, \mathrm{d}u\, \mathrm{d}v + \mathbb{N}\, \mathrm{d}v^2 = \lambda\, (\mathbb{E}\, \mathrm{d}u^2 + 2\, \mathbb{F}\, \mathrm{d}u\, \mathrm{d}v + \mathbb{G}\, \mathrm{d}v^2)$ everywhere on S. Therefore S must be part of a sphere. $\square$