Homework 4: The First Fundamental Form

(Total 20 pts + bonus 5 pts; Due Oct. 14 12pm)

Question 1. (5 pts) *Calculate the first fundamental form of the surface*

$$\sigma(u,v) = (3\sin u\cos v, 2\sin u\sin v, \cos u). \tag{1}$$

Solution. We have

$$\sigma_u = (3\cos u\cos v, 2\cos u\sin v, -\sin u),\ \ \sigma_v = (-3\sin u\sin v, 2\sin u\cos v, 0). \tag{2}$$

Therefore

$$\begin{aligned} \mathbb{E}(u,v) &= 5\cos^2 u\cos^2 v + 3\cos^2 u + 1, &(3)\\ \mathbb{F}(u,v) &= -5\cos u\sin u\cos v\sin v, &(4)\\ \mathbb{G}(u,v) &= 5\sin^2 u\sin^2 v + 4\sin^2 u. &(5) \end{aligned}$$

Therefore the first fundamental form is

$$(5\cos^2 u\cos^2 v + 3\cos^2 u + 1)\,\mathrm{d}u^2 - 10\cos u\sin u\cos v\sin v\,\mathrm{d}u\,\mathrm{d}v + (5\sin^2 u\sin^2 v + 4\sin^2 u)\,\mathrm{d}v^2. \tag{6}$$

Question 2. (5 pts) *Consider the surface patch $\sigma(u,v) = (u\cos v, u\sin v, \ln(\cos v) + u)$. Let $u_1 < u_2$ be arbitrary. Show that the arc length of the curve $\sigma(t,v)$ between $t = u_1$ and $t = u_2$ is independent of v.*

Proof. We calculate the first fundamental form:

$$\sigma_u = (\cos v, \sin v, 1), \qquad \sigma_v = \left(-u\sin v, u\cos v, -\frac{\sin v}{\cos v}\right) \tag{7}$$

and consequently

$$\begin{aligned} \mathbb{E}(u,v) &= 2, &(8)\\ \mathbb{F}(u,v) &= -\frac{\sin v}{\cos v}, &(9)\\ \mathbb{G}(u,v) &= u^2 + \frac{\sin^2 v}{\cos^2 v}. &(10) \end{aligned}$$

Now the arc length of $\sigma(t,v)$ between $t = u_1$ and $t = u_2$ is given by

$$\begin{aligned} L &= \int_{u_1}^{u_2} \langle \sigma_u, \sigma_u\rangle_{\sigma(t,v),S}^{1/2}\,\mathrm{d}t \\ &= \int_{u_1}^{u_2} \sqrt{\mathbb{E}(t,v)\cdot 1^2 + 2\,\mathbb{F}(t,v)\cdot 1\cdot 0 + \mathbb{G}(t,v)\cdot 0^2}\,\mathrm{d}t \\ &= \sqrt{2}\,(u_2 - u_1) &(11) \end{aligned}$$

which is clearly independent of v. □

QUESTION 3. (10 PTS) *Let the first fundamental form for a surface patch be* $\mathrm{d}u^2 + (1 + u^2)\,\mathrm{d}v^2$.

a) (8 PTS) *Calculate the lengths of the three sides and the three angles of the curvilinear triangle bounded by images of* $u = \frac{v^2}{2}, u = -\frac{v^2}{2}, v = 1$.

b) (2 PTS) *Prove that the area of the curvilinear triangle is greater than* $1/3$.

Solution.

a) We first parametrize the three see curves in the u-v plane:

- $u = \frac{v^2}{2}$: $(u_1(t), v_1(t))$ with $u_1(t) = \frac{t^2}{2}, v_1(t) = t$;
- $u = -\frac{v^2}{2}$: $(u_2(t), v_2(t))$ with $u_2(t) = -\frac{t^2}{2}, v_2(t) = t$;
- $v = 1$: $(u_3(t), v_3(t))$ with $u_3(t) = t, v_3(t) = 1$.

The three vertices are given by

- $V_1 = \left(-\frac{1}{2}, 1\right)$: Intersection of $(u_2(t), v_2(t))$ with $(u_3(t), v_3(t))$.
- $V_2 = \left(\frac{1}{2}, 1\right)$: Intersection of $(u_1(t), v_1(t))$ with $(u_3(t), v_3(t))$.
- $V_3 = (0, 0)$: Intersection of $(u_1(t), v_1(t))$ with $(u_2(t), v_2(t))$.

Now we are ready to calculate:

- The $(u_1(t), v_1(t))$ side between V_2, V_3. Here we have $(0, 0) = (u_1(0), v_1(0))$ and $\left(\frac{1}{2}, 1\right) = (u_1(1), v_1(1))$. Therefore the arc length is given by

$$\begin{aligned} L_1 &= \int_0^1 \sqrt{1 \cdot t^2 + \left(1 + \frac{t^4}{4}\right) \cdot 1^2}\,\mathrm{d}t \\ &= \int_0^1 \frac{t^2 + 2}{2}\,\mathrm{d}t = \frac{7}{6}. \end{aligned} \tag{12}$$

- The $(u_2(t), v_2(t))$ side between V_1,V_3. Here we have $(0, 0) = (u_2(0), v_2(0))$ and $\left(-\frac{1}{2}, 1\right) = (u_2(1), v_2(1))$. Therefore

$$\begin{aligned} L_2 &= \int_0^1 \sqrt{1 \cdot (-t)^2 + \left(1 + \frac{t^4}{4}\right) \cdot 1^2}\,\mathrm{d}t \\ &= \frac{7}{6}. \end{aligned} \tag{13}$$

- The $(u_3(t),\ v_3(t))$ side between $V_1,\ V_2$. Here we have $\left(-\frac{1}{2},\ 1\right) = \left(u_3\left(-\frac{1}{2}\right), v_3\left(-\frac{1}{2}\right)\right)$ and $\left(\frac{1}{2}, 1\right) = \left(u_3\left(\frac{1}{2}\right), v_3\left(\frac{1}{2}\right)\right)$. Consequently

$$L_3 = \int_{-1/2}^{1/2} \sqrt{1 \cdot 1^2}\,\mathrm{d}t = 1. \tag{14}$$

- The angle A_1 at V_1. We have $V_1 = \left(-\frac{1}{2}, 1\right) = (u_2(1), v_2(1)) = \left(u_3\left(-\frac{1}{2}\right), v_3\left(-\frac{1}{2}\right)\right)$. Thus we calculate

$$\mathbb{E}(V_1) = 1, \mathbb{F}(V_1) = 0, \mathbb{G}(V_1) = \frac{5}{4} \tag{15}$$

and

$$(u_2'(1), v_2'(1)) = (-1, 1), \qquad (u_3'(1), v_3'(1)) = (1, 0). \tag{16}$$

Therefore

$$\cos A_1 = \frac{1 \cdot (-1) \cdot 1 + \frac{5}{4} \cdot 1 \cdot 0}{\sqrt{1 \cdot (-1)^2 + \frac{5}{4} 1^2} \sqrt{1 \cdot 1^2 + \frac{5}{4} \cdot 0^2}} = -\frac{2}{3}. \tag{17}$$

Note that in fact this is the outer angle, so the inner angle should be $\cos A_1 = \frac{2}{3}$.

- The angle A_2 at V_2. We have $V_2 = \left(\frac{1}{2}, 1\right) = (u_1(1), v_1(1)) = \left(u_3\left(\frac{1}{2}\right), v_3\left(\frac{1}{2}\right)\right)$. Thus we calculate

$$\mathbb{E}(V_2) = 1, \mathbb{F}(V_2) = 0, \mathbb{G}(V_2) = \frac{5}{4} \tag{18}$$

and

$$(u_1'(1), v_1'(1)) = (1, 1), \qquad (u_3'(1), v_3'(1)) = (1, 0). \tag{19}$$

Therefore

$$\cos A_2 = \frac{2}{3}. \tag{20}$$

- The angle A_3 at V_3. We have $V_3 = (0, 0) = (u_1(0), v_1(0)) = (u_2(0), v_2(0))$. Thus we calculate

$$\mathbb{E}(V_3) = 1, \mathbb{F}(V_3) = 0, \mathbb{G}(V_3) = 1 \tag{21}$$

and

$$(u_1'(0), v_1'(0)) = (0, 1), \qquad (u_2'(0), v_2'(0)) = (0, 1) \tag{22}$$

which means

$$\cos A_3 = 1. \tag{23}$$

b) We have

$$\begin{aligned} A &= \int_U \sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\, \mathrm{d}u\, \mathrm{d}v \\ &= \int_U \sqrt{1 + u^2}\, \mathrm{d}u\, \mathrm{d}v \\ &\geqslant \int_U \mathrm{d}u\, \mathrm{d}v \\ &= \int_0^1 \left[\int_{-v^2/2}^{v^2/2} \mathrm{d}u \right] \mathrm{d}v = \frac{1}{3}. \end{aligned} \tag{24}$$

The following are more abstract or technical questions. They carry bonus points.

QUESTION 4. (**Bonus, 5 pts**) *Consider the surface of revolution $\sigma(u, v) = (f(u) \cos v, f(u) \sin v, u)$ where $f(u) > 0$ and $v \in [0, 2\pi]$.*

a) (2 PTS) *Prove that it can always be parametrized so that the first fundamental form becomes $\mathbb{E}(v)\, \mathrm{d}u^2 + \mathrm{d}v^2$.*

b) (2 PTS) *Find a conformal mapping between $\sigma(u,v)$ and the plane.*

c) (1 PT) *For what f is such a surface developable? Justify your claim.*

Proof.

a) We calculate its first fundamental form:

$$\sigma_u = (f'(u)\cos v,\, f'(u)\sin v, 1), \qquad \sigma_v = (-f(u)\sin v,\, f(u)\cos v, 0) \tag{25}$$

which give

$$\mathbb{E} = 1 + f'(u)^2, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = f(u)^2. \tag{26}$$

The first fundamental form is then

$$(1 + f'(u)^2)\,\mathrm{d}u^2 + f(u)^2\,\mathrm{d}v^2. \tag{27}$$

Now we set $\tilde{v} = F(u) := \int \sqrt{1 + f'(u)^2}$ and $\tilde{u} = v$ which gives

$$\mathrm{d}\tilde{v} = \sqrt{1 + f'(u)^2}\,\mathrm{d}u, \qquad \mathrm{d}\tilde{u} = \mathrm{d}v \tag{28}$$

which gives the new first fundamental form

$$\tilde{E}(\tilde{v})\,\mathrm{d}\tilde{u}^2 + \mathrm{d}\tilde{v}^2 \tag{29}$$

where

$$\tilde{E}(\tilde{v}) = f(F^{-1}(\tilde{v})) \tag{30}$$

with F^{-1} the inverse function of F.

b) From (27) we see that, if we set $G(u) := \int \frac{\sqrt{1 + f'(u)^2}}{f(u)}\,\mathrm{d}u$, and then $\tilde{u} = G(u), \tilde{v} = v$, we would have (27) becomes

$$f(G^{-1}(\tilde{u}))^2\,[\mathrm{d}\tilde{u}^2 + \mathrm{d}\tilde{v}^2] \tag{31}$$

so $(\tilde{u}, \tilde{v})$ gives a conformal mapping between the surface and the plane.

c) For S to be developable, it must be ruled, and along the straight lines the normal does not change. We calculate

$$\sigma_u \times \sigma_v = f(u)\,(-\cos v, \sin v,\, f'(u)). \tag{32}$$

Now fix (u_0, v_0). We look for those (u, v) very close to (u_0, v_0) such that

$$(-\cos v_0, \sin v_0,\, f'(u_0)) \parallel (-\cos v, \sin v,\, f'(u)). \tag{33}$$

Calculating the cross product we see that necessarily $\sin(v - v_0) = 0 \Longrightarrow v - v_0 = k\,\pi$ for $k \in \mathbb{Z}$. As we are considering (u, v) close to (u_0, v_0), there must hold $v = v_0$. Therefore $\sigma(v_0,\, u)$ must be a straight line. Consequently $f(u)$ is a linear function. There are two cases:

i. $f(u)$ is constant. The surface is a cylinder.

ii. $f(u)$ is linear but not constant. The surface is a cone. $\square$