Homework 3: Differential Geometry of Curves

(Total 20 pts + bonus 5 pts; Due Sept. 30 12pm)

Question 1. (10 pts) *Calculate* T, N, B, κ, τ *of the curve* $x(t) = (t, t^2, t^4)$ *at the point* $(1, 1, 1)$.

Question 2. (5 pts) *Let* f *be a smooth function. Calculate the curvature and the torsion of the curve that is the intersection of* $x = y$ *and* $z = f(x)$.

Question 3. (5 pts) *Let* $x(s)$ *be a curve with arc length parametrization, and satisfies* $\|x(s)\| \leqslant \|x(s_0)\| \leqslant 1$ *for all* s *sufficiently close to* x_0. *Prove* $\kappa(s_0) \geqslant 1$. *(Hint: Consider* $f(s) = \|x(s)\|^2$. *Then* $f(s)$ *has a local maximum at* s_0. *Calculate* $f''(s_0)$*)*

The following are more abstract or technical questions. They carry bonus points.

Question 4. (**Bonus, 5 pts**) *Let* $x(t)$ *be a smooth* ***plane curve****. Assume that the chord length between* $x(t_1), x(t_2)$ *depends only on* $|t_1 - t_2|$ *for all* $t_1, t_2 \in (\alpha, \beta)$, *that is there is some function* F *such that* $\|x(t_1) - x(t_2)\| = F(|t_1 - t_2|)$ *for all* $t_1, t_2 \in (\alpha, \beta)$. *Prove that* $x(t)$ *is part of either a circle or a straightline. (Hint: First from* $\|x(t + \delta t) - x(t)\| = F(\delta t)$ *show that* $\|x'(t)\| =$ *constant for every* t. *Next apply Taylor expansion to* $\|x(t + \delta t) - x(t)\|^2 = F(\delta t)^2$ *to reach the conclusion.)*