

HOMEWORK 1: PREREQUISITES

(Total 20 pts + bonus pts; Due Sept. 16 12pm)

QUESTION 1. (5 PTS) Let $f, g: \mathbb{R} \mapsto \mathbb{R}^3$ be differentiable. Prove

- i. $(f(t) \cdot g(t))' = f'(t) \cdot g(t) + f(t) \cdot g'(t);$
- ii. $(f(t) \times g(t))' = f'(t) \times g(t) + f(t) \times g'(t);$

QUESTION 2. (5 PTS) Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$ be defined as $f(x) := (Ax) \cdot x$, where $A = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$. Calculate the Taylor expansion of f to the second order at $(1, 1)$.

QUESTION 3. (5 PTS) Let $f: \mathbb{R} \mapsto \mathbb{R}$ be differentiable and non-negative with $f(0) = 1$. Further assume

$$f'(t) \leq t f(t) \tag{1}$$

for all $t \geq 0$. Prove that

$$f(t) \leq \exp(t^2/2) \tag{2}$$

for all $t \geq 0$.

QUESTION 4. (5 PTS) Let $f(t): \mathbb{R} \mapsto \mathbb{R}^3$ be nonzero and smooth¹. Then $\|f(t)\|$ is a constant $\iff f'(t) \cdot f(t) = 0$.

Note. To prove $A \iff B$, you need to prove two things:

- \implies : if statement A is true then statement B is true;
- \impliedby : if statement B is true then statement A is true.

The following are more abstract or technical questions. They carry bonus points.

QUESTION 5. (**BONUS, 5 PTS**) Let $f(t): \mathbb{R} \mapsto \mathbb{R}^3$ be nonzero and smooth. Then

- i. (2 PTS) $f(t)$ has fixed direction $\iff f(t) \times f'(t) = 0;$
- ii. (3 PTS) $f(t) \perp v$ for some constant vector $v \iff (f(t) \times f'(t)) \cdot f''(t) = 0.$

1. In 348 “smooth” means there is no need to prove differentiability or integrability, no matter how many derivatives or integrals are being taken.