Review for Final: Theory of Surfaces

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Suggestion: preparation for the final.

1. Go through lecture notes;
2. Re-do the two midterms;
3. Re-do all homeworks;
4. Go through textbook and work on exercises in it.

1. Q&A

- Signed curvature for plane curves.

  It is exactly the geodesic curvature of the curve. Note that the normal curvature of any plane curve is zero.

- Q2 of HW7.

  QUESTION. *Let γ be a curve on S. Let $w, \tilde{w}$ be unit vector fields along γ. Further assume that at every $p \in \gamma$, there holds the angle between w, $\tilde{w}$, $\angle(w, \tilde{w}) = \theta_0$, a constant. Prove or disprove: w is parallel along γ if and only if $\tilde{w}$ is parallel along γ.*

  Solution. The claim is true. We parametrize γ by some $x(t)$ and simply write $w(t)$, $\tilde{w}(t)$. We discuss two cases.

  1. $\angle(w, \tilde{w}) = 0$ or π. Then $\tilde{w} = w$ or $-w$. Clearly $\nabla_\gamma \tilde{w} = 0$.
  2. Otherwise. Notice that this means $\{w, \tilde{w}\}$ for a basis for the tangent plane. By assumption we have $w \cdot \tilde{w} = \text{constant}$. Therefore

  $$w' \cdot \tilde{w} + w \cdot \tilde{w}' = 0. \tag{1}$$

  Since $\nabla_\gamma w = 0$, we have $w' \perp \tilde{w}$. Therefore $\tilde{w}' \cdot w = 0$. On the other hand, as $\|\tilde{w}\| = 1$ we have $\tilde{w}' \cdot \tilde{w} = 0$. Thus $\tilde{w} \parallel N$ and consequently $\nabla_\gamma \tilde{w} = 0$.

- How to show a curve is geodesic?
  - A curve γ is a geodesic when its unit tangent vector stays parallel:

  $$\nabla_\gamma T = 0. \tag{2}$$

  - Thus if $T = \alpha\, \sigma_u + \beta\, \sigma_v$, $\nabla_\gamma T = 0$ becomes

  $$\alpha' + (\alpha, \beta)\, (\Gamma_{ij}^1) \begin{pmatrix} u' \\ v' \end{pmatrix} = 0, \qquad \beta' + (\alpha, \beta)\, (\Gamma_{ij}^2) \begin{pmatrix} u' \\ v' \end{pmatrix} = 0. \tag{3}$$

  - Let γ be given as $\sigma(u(t), v(t))$.
    - Case 1. t is arc length. Then γ is a geodesic if and only if

  $$u'' + (u', v')\, (\Gamma_{ij}^1) \begin{pmatrix} u' \\ v' \end{pmatrix} = 0, \qquad v'' + (u', v')\, (\Gamma_{ij}^2) \begin{pmatrix} u' \\ v' \end{pmatrix} = 0. \tag{4}$$

- Case 2. t may not be arc length. Then calculate

$$T=\frac{u'}{\sqrt{\mathbb{E}\,u'^2+2\,\mathbb{F}\,u'\,v'+\mathbb{G}\,v'^2}}\,\sigma_u+\frac{v'}{\sqrt{\mathbb{E}\,u'^2+2\,\mathbb{F}\,u'\,v'+\mathbb{G}\,v'^2}}\,\sigma_v. \tag{5}$$

Thus $\nabla_\gamma T=0$ becomes

$$\left(\frac{u'}{\sqrt{\mathbb{E}\,u'^2+2\,\mathbb{F}\,u'\,v'+\mathbb{G}\,v'^2}}\right)'+\frac{(u',v')(\Gamma_{ij}^1)\begin{pmatrix}u'\\v'\end{pmatrix}}{\sqrt{\mathbb{E}\,u'^2+2\,\mathbb{F}\,u'\,v'+\mathbb{G}\,v'^2}}=0, \tag{6}$$

$$\left(\frac{v'}{\sqrt{\mathbb{E}\,u'^2+2\,\mathbb{F}\,u'\,v'+\mathbb{G}\,v'^2}}\right)'+\frac{(u',v')(\Gamma_{ij}^2)\begin{pmatrix}u'\\v'\end{pmatrix}}{\sqrt{\mathbb{E}\,u'^2+2\,\mathbb{F}\,u'\,v'+\mathbb{G}\,v'^2}}=0. \tag{7}$$

- Q3 of Midterm 2.

QUESTION. *Consider the same surface patch as in Questions 1 and 2, $\sigma(u,v):=(u,v,e^{uv})$.*

a) (3 PTS) *Calculate the Christoffel symbols Γ_{ij}^k.*

b) (2 PTS) *Is $u=0$ a geodesic? Justify your claim.*

Solution.

a) We calculate

$$\sigma_u\times\sigma_v=(-e^{uv}\,v,-e^{uv}\,u,1). \tag{8}$$

Therefore

$$\begin{pmatrix}0\\0\\e^{uv}\,v^2\end{pmatrix}=\sigma_{uu}=\Gamma_{11}^1\begin{pmatrix}1\\0\\e^{uv}\,v\end{pmatrix}+\Gamma_{11}^2\begin{pmatrix}0\\1\\e^{uv}\,u\end{pmatrix}+l\begin{pmatrix}-e^{uv}\,v\\-e^{uv}\,u\\1\end{pmatrix}. \tag{9}$$

We see that $\Gamma_{11}^1=l\,e^{uv}\,v,\Gamma_{11}^2=l\,e^{uv}\,u$. Substituting into the third equation we have

$$e^{uv}\,v^2=l\,e^{2uv}\,v^2+l\,e^{2uv}\,u^2+l\Longrightarrow l=\frac{e^{uv}\,v^2}{e^{2uv}\,(u^2+v^2)+1}. \tag{10}$$

Therefore

$$\Gamma_{11}^1=\frac{e^{2uv}\,v^3}{e^{2uv}\,(u^2+v^2)+1},\qquad\Gamma_{11}^2=\frac{e^{2uv}\,v^2\,u}{e^{2uv}\,(u^2+v^2)+1}. \tag{11}$$

Next we have

$$\begin{pmatrix}0\\0\\e^{uv}\,(1+u\,v)\end{pmatrix}=\sigma_{uv}=\Gamma_{12}^1\begin{pmatrix}1\\0\\e^{uv}\,v\end{pmatrix}+\Gamma_{12}^2\begin{pmatrix}0\\1\\e^{uv}\,u\end{pmatrix}+m\begin{pmatrix}-e^{uv}\,v\\-e^{uv}\,u\\1\end{pmatrix} \tag{12}$$

which gives

$$\Gamma_{12}^1=\frac{e^{2uv}\,(1+u\,v)\,v}{e^{2uv}\,(u^2+v^2)+1},\qquad\Gamma_{12}^2=\frac{e^{2uv}\,(1+u\,v)\,u}{e^{2uv}\,(u^2+v^2)+1}. \tag{13}$$

Finally we calculate

$$\begin{pmatrix} 0 \\ 0 \\ e^{uv} u^2 \end{pmatrix} = \sigma_{vv} = \Gamma_{22}^1 \begin{pmatrix} 1 \\ 0 \\ e^{uv} v \end{pmatrix} + \Gamma_{22}^2 \begin{pmatrix} 0 \\ 1 \\ e^{uv} u \end{pmatrix} + n \begin{pmatrix} -e^{uv} v \\ -e^{uv} u \\ 1 \end{pmatrix} \tag{14}$$

which gives

$$\Gamma_{22}^1 = \frac{e^{2uv} v u^2}{e^{2uv} (u^2 + v^2) + 1}, \qquad \Gamma_{22}^2 = \frac{e^{2uv} u^3}{e^{2uv} (u^2 + v^2) + 1}. \tag{15}$$

b) We parametrize $u = 0$ as $u(t) = 0, v(t) = t$. Note that $x(t) := \sigma(u(t), v(t)) = (0, t, 1)$ is arc length parametrized.

Next along $u = 0$, we have

$$\begin{aligned} \Gamma_{11}^1 &= \frac{v^3}{1 + v^2}, & \Gamma_{11}^2 &= 0, \\ \Gamma_{12}^1 &= \frac{v}{1 + v^2}, & \Gamma_{12}^2 &= 0, \\ \Gamma_{22}^1 &= 0, & \Gamma_{22}^2 &= 0. \end{aligned} \tag{16}$$

Therefore the geodesic equations are satisfied along $u = 0$:

$$0 + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{v^3}{1+v^2} & 0^1 \\ 0^1 & 0^1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \tag{17}$$

$$0 + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \tag{18}$$

So $u = 0$ is a geodesic.

- Developable surface.
  - Intuition. A surface that can be "flattened" without stretching or squeezing. A surface that can have a faithful plane map.
  - How to check? $K = 0$.
- Homework 9.

QUESTION 1. (5 PTS) *Let S be a regular, orientable, compact surface with positive Gaussian curvature: $K > K_{\min} > 0$. Prove that the surface area of S is less than $4\pi / K_{\min}$.*

Proof. Take any simple closed curve γ on S. γ divides S into two regions Ω_1, Ω_2. Let γ be oriented such that Ω_1 is its interior. Then by Gauss-Bonnet theorem

$$\int_{\Omega_1} K \, \mathrm{d}S + \int_\gamma \kappa_g \, \mathrm{d}s = 2\pi, \qquad \int_{\Omega_2} K \, \mathrm{d}S + \int_{-\gamma} \kappa_g \, \mathrm{d}s = 2\pi \tag{19}$$

where $-\gamma$ is γ with the opposite orientation. Since

$$\int_{-\gamma} \kappa_g \, \mathrm{d}s = -\int_\gamma \kappa_g \, \mathrm{d}s \tag{20}$$

we have

$$4\,\pi = \int_S K\,\mathrm{d}S \geqslant \int_S K_{\min}\,\mathrm{d}S \tag{21}$$

and the conclusion follows. □

QUESTION 2. (5 PTS) *Let S be a compact oriented surface that can be smoothly deformed into a sphere. Let γ be a simple closed geodesic separating S into two regions A, B. Let $\mathcal{G}: S \mapsto \mathbb{S}^2$ be the Gauss map. Prove that $\mathcal{G}(A)$ and $\mathcal{G}(B)$ have the same area.*

Proof. Since $\mathbb{S}^2$ taking away one point can be covered by one single surface patch, so can S. Let $\sigma(u,v)$ be such a surface patch for S. Then we have

$$\int_S K\,\mathrm{d}S = \int_U K(u,v)\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v. \tag{22}$$

Now let U_A, U_B be such that $\sigma(U_A) = A, \sigma(U_B) = B$ (maybe missing one point). Denote $N(u,v) := \mathcal{G}(\sigma(u,v))$. Then we have

$$\text{Area of } \mathcal{G}(A) = \int_{U_A} \|N_u \times N_v\|\,\mathrm{d}u\,\mathrm{d}v. \tag{23}$$

Recalling

$$-N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, \qquad -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v, \tag{24}$$

we have

$$N_u \times N_v = \det\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \sigma_u \times \sigma_v = K\,\sigma_u \times \sigma_v. \tag{25}$$

Consequently

$$\begin{aligned} \int_{U_A} \|N_u \times N_v\|\,\mathrm{d}u\,\mathrm{d}v &= \int_{U_A} K\,\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{U_A} K(u,v)\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_A K\,\mathrm{d}S \\ &= 2\,\pi - \int_\gamma \kappa_g\,\mathrm{d}s = 2\,\pi. \end{aligned} \tag{26}$$

Similarly we have Area of $\mathcal{G}(B) = 2\,\pi$. □

QUESTION 3. *Let S be a developable surface. Let γ be a curve on S. Let $\tilde{\gamma}$ be the curve corresponding to γ on the plane that is the "flattened" S. Prove or disprove: The geodesic curvature of γ and the signed curvature of $\tilde{\gamma}$ are the same at corresponding points.*

Solution. We prove that the claim is true.

Let $\sigma(u,v): U \mapsto S$ a local isometry from the plane to S. Clearly $\sigma(u,v)$ can serve as a surface patch. Furthermore we have $\mathbb{E} = \mathbb{G} = 1, \mathbb{F} = 0$ and consequently all $\Gamma_{ij}^k = 0$. Note that this implies the surface normal

$$N = \sigma_u \times \sigma_v, \tag{27}$$

and that $\sigma_{uu}, \sigma_{uv}, \sigma_{vv} \parallel N$.

Now let $(u(s),\ v(s))$ be an arc length parametrization of $\tilde{\gamma}$. We then see that $x(s) := \sigma(u(s), v(s))$ is an arc length parametrization of γ. Thus

$$\begin{aligned}\kappa_g &= x'' \cdot (N \times x') \\ &= [\sigma_{uu}(u')^2 + 2\,\sigma_{uv}\,u'\,v' + \sigma_{vv}(v')^2 + \sigma_u\,u'' + \sigma_v\,v''] \cdot [(\sigma_u \times \sigma_v) \cdot (u'\,\sigma_u + v'\,\sigma_v)] \\ &= [\sigma_{uu}(u')^2 + 2\,\sigma_{uv}\,u'\,v' + \sigma_{vv}(v')^2 + \sigma_u\,u'' + \sigma_v\,v''] \cdot (u'\,\sigma_v - v'\,\sigma_u) \\ &= v''\,u' - u''\,v' \\ &= \begin{pmatrix} u \\ v \end{pmatrix}'' \cdot \left[\begin{pmatrix} u \\ v \end{pmatrix}'\right]^{\perp} = \kappa_s. \end{aligned} \tag{28}$$

Question 4. (5 pts) *Let $f\colon S_1 \mapsto S_2$ be a local isometry. Let a curve $\gamma_1 \subset S_1$ and $\gamma_2 := f(\gamma_1)$. Let $w_1(s)$ be a parallel tangent vector field along γ_1. For every $p \in \gamma_1$, Let $w_2(f(p)) := (D f)(p)(w_1(p))$. Then $w_2(s)$ is a tangent vector field along γ_2. Prove or disprove: w_2 is parallel along γ_2.*

Solution. We prove that the claim is true.

Let $\sigma_1(u,v)$ be a surface patch for S_1 and let $\sigma_2(u,v) := f(\sigma_1(u,v))$. Also let $x_1(s)$ be an arc length parametrization of γ_1 and let $x_2(s) := f(x_1(s))$. Since f is a local isometry, s is also the arc length parameter of γ_2.

In this setup we have $\sigma_{2,u} = (D f)(\sigma_{1,u})$ and $\sigma_{2,v} = (D f)(\sigma_1, v)$. Now let $w_1(s) = \alpha(s)\,\sigma_{1,u} + \beta(s)\,\sigma_{1,v}$. Then we have $w_2(s) = \alpha(s)\,\sigma_{2,u} + \beta(s)\,\sigma_{2,v}$. Since $w_1(s)$ is parallel along γ_1, we have

$$\begin{aligned}(\mathbb{E}_1\,\alpha + \mathbb{F}_1\,\beta)' &= \frac{1}{2}\begin{pmatrix} \alpha & \beta \end{pmatrix}\begin{pmatrix} \mathbb{E}_1 & \mathbb{F}_1 \\ \mathbb{F}_1 & \mathbb{G}_1 \end{pmatrix}_u \begin{pmatrix} u' \\ v' \end{pmatrix}, \\ (\mathbb{F}_1\,\alpha + \mathbb{G}_1\,\beta)' &= \frac{1}{2}\begin{pmatrix} \alpha & \beta \end{pmatrix}\begin{pmatrix} \mathbb{E}_1 & \mathbb{F}_1 \\ \mathbb{F}_1 & \mathbb{G}_1 \end{pmatrix}_v \begin{pmatrix} u' \\ v' \end{pmatrix}. \end{aligned} \tag{29}$$

But since $\mathbb{E}_1 = \mathbb{E}_2, \mathbb{F}_1 = \mathbb{F}_2, \mathbb{G}_1 = \mathbb{G}_2$, $(\alpha(s), \beta(s))$ satisfies the corresponding equations on S_2 and consequently w_2 is also parallel along γ_2.