Midterm Review III: Cauchy Problems and Wave Equations

In this last review lecture we will first clarify some steps in solving Cauchy problems for 1st order quasi-linear PDEs, and then summarize what we have covered regarding wave equations.

1. Cauchy problem for 1st order PDEs.

We will focus on the quasi-linear case which usually yields general solutions in an implicit form. We illustrate the strategy through the following example.

Example 1. (§2.8, 15) Find the solution surface of the equation

$$\left(u^2-y^2\right) u_x + x\, y\, u_y + x\, u = 0, \qquad u = y = x,\ \ x > 0. \tag{1}$$

Solution. First we rewrite the equation in the form we have been dealt with:

$$\left(u^2-y^2\right) u_x + x\, y\, u_y = -\, x\, u, \qquad u = x \text{ along } y = x. \tag{2}$$

The characteristics equation is

$$\frac{\mathrm{d}x}{u^2-y^2} = \frac{\mathrm{d}y}{x\, y} = \frac{\mathrm{d}u}{-\, x\, u}. \tag{3}$$

Using the 2nd and the 3rd ratios we obtain the ratio

$$\frac{u\, \mathrm{d}y + y\, \mathrm{d}u}{u\, x\, y + (-\, x\, u)\, y} = \frac{\mathrm{d}(u\, y)}{0}. \tag{4}$$

For the second invariant, we observe that $u^2 - y^2 = (u+y)\,(u-y)$ while

$$\frac{\mathrm{d}y}{x\, y} = \frac{\mathrm{d}u}{-\, x\, u} \implies \frac{\mathrm{d}(u+y)}{-\, x\, (u-y)}. \tag{5}$$

Thus we have

$$\frac{\mathrm{d}x}{u^2-y^2} = \frac{\mathrm{d}(u+y)}{-\, x\, (u-y)} \implies \frac{\mathrm{d}x}{u+y} = -\frac{\mathrm{d}(u+y)}{x} \implies \frac{\mathrm{d}\left[x^2 + (u+y)^2\right]}{0}. \tag{6}$$

Now that we have two invariants (two ratios with 0 denominators), the general solution can be written as

$$F\left(u\, y, x^2 + (u+y)^2\right) = 0. \tag{7}$$

for an arbitrary function $F(\phi, \psi)$.

To solve the Cauchy problem, we substitute u by x and y by x, and obtain

$$F\left(x^2, 5\, x^2\right) = 0. \tag{8}$$

Our task is to find one particular F such that this is true for all x. One universal approach (as usual, an universal approach may not be the most clever or efficient one) is the following.

Let $\phi = x^2$. Then we can solve $x = \sqrt{\phi}$. Substituting this into $5\, x^2$ gives $5\, x^2 = 5\, \phi$. Thus we can choose

$$F(\phi, \psi) = \psi - 5\, \phi. \tag{9}$$

Now replacing ϕ by $u\, y$ and ψ by $x^2 + (u+y)^2$, we obtain the solution (implicitly)

$$x^2 + (u+y)^2 - 5\, u\, y = x^2 + u^2 + y^2 - 3\, u\, y = 0. \tag{10}$$

Summary. In general, when solving Cauchy problems for quasi-linear 1st order PDEs, the last step is to find out one F such that

$$F(f(x), g(x)) = 0 \tag{11}$$

for all x. The most mechanical way to determine such F is the following.

1. Let $\phi = f(x)$, invert f to obtain $x = h(\phi)$.
2. Substitute $g(x) = g(h(\phi))$.
3. Set $F(\phi, \psi) = \psi - g(h(\phi))$.

2. Wave equations.

The basic requirement is to be able to find the solution through applying the correct formula based on the types of boundary and initial values of the problem. We list different types of problems and the corresponding formulas below.

These formulas will not be provided in the exam. Familiarity of how they are derived will help remembering them.

- Initial value problem.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad -\infty < x < \infty, \ t > 0 && (12)\\ u(x,0) &= f(x), \qquad -\infty < x < \infty && (13)\\ u_t(x,0) &= g(x), \qquad -\infty < x < \infty && (14) \end{aligned}$$

  The solution is (d'Alembert's formula)

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y. \tag{15}$$

  How to derive: Use the formula for general solutions.

- Non-homogeneous initial value problem.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), \qquad -\infty < x < \infty, \quad t > 0 && (16)\\ u(x,0) &= f(x), \qquad -\infty < x < \infty && (17)\\ u_t(x,0) &= g(x), \qquad -\infty < x < \infty && (18) \end{aligned}$$

  The solution is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y + \frac{1}{2c}\iint_{\Omega(x,t)} h(y,s)\,\mathrm{d}y\,\mathrm{d}s \tag{19}$$

  where

$$\Omega(x,t) = \{(y,s)\colon \ s > 0, \ y \in (x - c(t-s), x + c(t-s))\}. \tag{20}$$

  How to derive: Green's formula, integrating the equation over $\Omega(x,t)$.

- Initial-boundary value problem I.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < \infty, \quad t > 0 && (21)\\ u(x,0) &= f(x), \qquad 0 \leqslant x < \infty, && (22)\\ u_t(x,0) &= g(x), \qquad 0 \leqslant x < \infty, && (23)\\ u(0,t) &= 0, \qquad 0 \leqslant t < \infty. && (24) \end{aligned}$$

  The solution is

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y & x > ct \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)\,\mathrm{d}y & 0 \leqslant x < ct \end{cases}. \tag{25}$$

  How to derive:

  - Method 1: Use the formula for general solutions.
  - Method 2: Extend f, g oddly and use d'Alembert's formula.

- Initial-boundary value problem II.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < \infty, \quad t > 0 && (26)\\ u(x,0) &= f(x), \quad 0 \leqslant x < \infty, && (27)\\ u_t(x,0) &= g(x), \quad 0 \leqslant x < \infty, && (28)\\ u_x(0,t) &= 0, \qquad 0 \leqslant t < \infty. && (29) \end{aligned}$$

The solution is

$$u(x,t)=\begin{cases}\frac{1}{2}\left[f(x+c\,t)+f(x-c\,t)\right]+\frac{1}{2\,c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y & x>c\,t\\ \frac{1}{2}\left[f(x+c\,t)+f(c\,t-x)\right]+\frac{1}{2\,c}\left[\int_0^{x+ct} g(y)\,\mathrm{d}y+\int_0^{ct-x} g(y)\,\mathrm{d}y\right] & 0\leqslant x<c\,t\end{cases}. \tag{30}$$

How to derive:

- Method 1: Use the formula for general solutions.
- Method 2: Extend f, g evenly and use d'Alembert's formula.

• Initial-boundary value problem III.

$$\begin{aligned} u_{tt}-c^2\,u_{xx} &= 0, \qquad 0<x<l, \quad t>0 && (31)\\ u(x,0) &= f(x), \quad 0\leqslant x<l, && (32)\\ u_t(x,0) &= g(x), \quad 0\leqslant x<l, && (33)\\ u(0,t) &= 0, \qquad 0\leqslant t<\infty && (34)\\ u(l,t) &= 0, \qquad 0\leqslant t<\infty. && (35) \end{aligned}$$

There is unfortunately no simple formula(s) for this one. We have to follow either of

- Method 1: Use the formula of general solution.
- Method 2: Extend f, g first evenly to $-l<x<l$, then extend them periodically with period $2\,l$ and then use d'Alembert's formula.