Math 337 2009 Final Review

As our final is cumulative, please do not forget to review material before the midterm.

Table of contents

How to prepare for the final: Some advice.

1. Work on homework problems without referring to the book. Then check against homework solutions.

2. Work on problems in "Study Guide" without the book at hand.

1. Method of characteristics.

See midterm reviews. The purpose of listing it here is to generate a complete "Table of Contents".

2. Method of separation of variables.

We illustrate this method using the following problem:

Example 1. (§10.13 2) Solve the Neumann problem

$$\nabla^2 u = 0, \qquad 0<x<1, \quad 0<y<1, \quad 0<z<1 \tag{1}$$

$$u_x(0,y,z) = 0, \tag{2}$$

$$u_x(1,y,z) = 0, \tag{3}$$

$$u_y(x,0,z) = 0, \tag{4}$$

$$u_y(x,1,z) = 0, \tag{5}$$

$$u_z(x,y,0) = \cos\pi x \cos\pi y, \tag{6}$$

$$u_z(x,y,1) = 0. \tag{7}$$

2.1. The idea.

The whole “separation of variables” method is based on the following naïve wishful thinking:

> The solution to a general boundary (initial, initial-boundary) value problem can be written as a sum of simple functions – products of single-variable functions – which satisfy the same equation and similar boundary conditions.

- Thanks to the Sturm-Liouville theory, this approach indeed works.
- To carry out this approach, one need the ability to
  i. Solve eigenvalue problems (for finding out these simple functions);
  ii. Expand a given function with respect to a set of eigenfunctions (for getting the final infinite sum formula).

2.2. The procedure.

The basic idea of the method of separation of variables is to represent the solution into the following form:

$$u(x,y,z)=\sum_{m,n} U_{mn}(x,y,z) \tag{8}$$

where each U_{mn} is of the simple form

$$U_{mn}(x,y,z)=X(x)\,Y(y)\,Z(z) \tag{9}$$

and is therefore relatively easy to find.

The basic assumption (guaranteed through the Sturm-Liouville theory) is that each U_{mn} is somewhat “independent” of others. In particular, each U_{mn} satisfies the same equation as their (alleged) sum u.

Under this assumption, we can try to determine each U_{mn} through the following steps.

1. **Separating the variables**.

   Recall that U_{mn} should satisfy the equation. This leads to

$$X''YZ+XY''Z+XYZ''=0. \tag{10}$$

The goal is to obtain equations for X, Y, Z individually. To do this, we need the help of the following property of functions:

> If
>
> $$f(x,y,z,\ldots)=g(x',y',z',\ldots) \tag{11}$$
>
> and if the variables $x,y,z,\ldots$ and all different from any of $x',y',z',\ldots$, then both f,g are constants.

To take advantage of this property we need to make the variables involved in the LHS different from those variables involved in the RHS. A typical technique is dividing the equation by U_{mn}. This leads to

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0. \tag{12}$$

Now we are free to move any term to the RHS. For example we move Z to the RHS to obtain

$$\frac{X''}{X}+\frac{Y''}{Y}=-\frac{Z''}{Z}. \tag{13}$$

As a consequence there is a constant λ such that

$$\frac{X''}{X}+\frac{Y''}{Y}=-\lambda, \qquad \frac{Z''}{Z}=\lambda. \tag{14}$$

Now we can do the same to the X,Y equation:

$$\frac{X''}{X}+\lambda=-\frac{Y''}{Y}. \tag{15}$$

The LHS involves only x and the RHS only y. Therefore there is another constant μ such that

$$\frac{X''}{X} + \lambda = -\mu, \qquad \frac{Y''}{Y} = \mu. \tag{16}$$

Thus we know have obtained equations for each of X, Y, Z, (very weakly) coupled together through two (to-be-determined) constants λ, μ.

2. **Impose boundary conditions**.

   To solve any of the equations for X, Y, Z we need boundary conditions. Recall that our purpose is to obtain the solution u in the form

$$u = \sum U_{mn}(x, y, z). \tag{17}$$

Thus the boundary conditions of u implies

$$u_x(0, y, z) = 0 \implies \sum U_{mn}(0, y, z)_x = 0; \tag{18}$$

$$u_x(1, y, z) = 0 \implies \sum U_{mn}(1, y, z)_x = 0; \tag{19}$$

$$u_y(x, 0, z) = 0 \implies \sum U_{mn}(x, 0, z)_y = 0; \tag{20}$$

$$u_y(x, 1, z) = 0 \implies \sum U_{mn}(x, 1, z)_y = 0; \tag{21}$$

$$u_z(x, y, 0) = \cos \pi x \cos \pi y \implies \sum U_{mn}(x, y, 0)_z = \cos \pi x \cos \pi y; \tag{22}$$

$$u_z(x, y, 1) = 0 \implies \sum U_{mn}(x, y, 1)_z = 0. \tag{23}$$

Remember that we would like to obtain boundary conditions for each U_{mn}. Now for the 0 boundary conditions, it is clear that the first thing to try is to require the same boundary conditions for U_{mn}, which in turn leads to definite boundary conditions for X, Y, Z:

$$X'(0)\, Y(y)\, Z(z) = U_{mn}(0, y, z)_x = 0 \implies X'(0) = 0; \tag{24}$$

$$U_{mn}(1, y, z)_x = 0 \implies X'(1) = 0; \tag{25}$$

$$U_{mn}(x, 0, z)_y = 0 \implies Y'(0) = 0; \tag{26}$$

$$U_{mn}(x, 1, z)_y = 0 \implies Y'(1) = 0; \tag{27}$$

$$U_{mn}(x, y, 1)_z = 0 \implies Z'(1) = 0. \tag{28}$$

However it is less clear what to do to the remaining one

$$\sum U_{mn}(x, y, 0)_z = \cos \pi x \cos \pi y \tag{29}$$

as it is easy to see that requiring

$$U_{mn}(x, y, 0)_z = \cos \pi x \cos \pi y \tag{30}$$

will surely not work.

3. **Solving eigenvalue problems**.

   We have the following problems to solve:

$$X'' + (\lambda + \mu)\, X = 0, \qquad X'(0) = X'(1) = 0; \tag{31}$$

$$Y'' - \mu\, Y = 0, \qquad Y'(0) = Y'(1) = 0; \tag{32}$$

$$Z'' - \lambda\, Z = 0, \qquad Z'(1) = 0, \quad \text{Condition at } z = 0 \text{ to be revealed.} \tag{33}$$

We compare: The problem for X involves two unknown constants; The problem fo Y involves one unknown constant; The problem for Z involves one unknown constant and is at the same time missing one boundary condition.

Solving $\boldsymbol{Y}$.

It is clear that we should start by solving Y.

$$Y'' - \mu\, Y = 0, \qquad Y'(0) = Y'(1) = 0; \tag{34}$$

This is an eigenvalue problem.

An *eigenvalue problem* is a differential equation, which involve a parameter λ, together with some homogeneous boundary conditions. Usually, for most values of λ no solution (except the trivial solution 0) exists. Those λ for which there are non-zero solutions are called *eigenvalues*, and these non-zero solutions are called the corresponding *eigenfunctions*.

To solve simple eigenvalue problems (those allow us to write down formulas for general solutions), the following steps are involved.

i. Writing down the general solutions of the equation.

For our equation $Y'' - \mu Y = 0$, the general solution has a different formula for each of the cases $\mu > 0, = 0, < 0$. We write

- $\mu > 0$:

$$Y(y) = A\, e^{\sqrt{\mu}\, y} + B^{-\sqrt{\mu}\, y}, \tag{35}$$

- $\mu = 0$:

$$Y(y) = A + B\, y \tag{36}$$

- $\mu < 0$:

$$Y(y) = A \cos\left(\sqrt{-\mu}\, y\right) + B \sin\left(\sqrt{-\mu}\, y\right). \tag{37}$$

ii. Finding out those μ which allows the problem to have non-zero solutions, these μ's are the eigenvalues. Since our formulas already satisfy the differential equation, all we need to do is to plug the formulas of the general solution into the boundary conditions and find out those μ for which A, B are not both forced to 0.

- $\mu > 0$. Boundary conditions become

$$\sqrt{\mu}\, A - \sqrt{\mu}\, B = 0, \qquad \sqrt{\mu}\, A\, e^{\sqrt{\mu}} - \sqrt{\mu}\, B\, e^{-\sqrt{\mu}} = 0 \tag{38}$$

which can be written as (after cancelling $\sqrt{\mu}$)

$$\begin{pmatrix} 1 & -1 \\ e^{\sqrt{\mu}} & -e^{-\sqrt{\mu}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{39}$$

We know from basic linear algebra that A, B can be not both 0 if and only if

$$\det \begin{pmatrix} 1 & -1 \\ e^{\sqrt{\mu}} & -e^{-\sqrt{\mu}} \end{pmatrix} = 0. \tag{40}$$

As the determinant is

$$e^{\sqrt{\mu}} - e^{-\sqrt{\mu}} \neq 0 \tag{41}$$

for all $\mu > 0$, we see that A, B have to both be 0. In other words, there is no $\mu > 0$ which allows nonzero solutions, or equivalently there is no positive eigenvalue.

- $\mu = 0$. Boundary conditions become

$$B = 0, \qquad B = 0. \tag{42}$$

We see that $Y = A$ the constant function is a solution. In other words, when $\mu = 0$ there are nonzero solutions. Equivalently, 0 is an eigenvalue, whose corresponding eigenfunctions are

$$Y = A \tag{43}$$

for any constant A.

- $\mu < 0$. Boundary conditions become

$$\sqrt{-\mu}\, B = 0, \qquad -\sqrt{-\mu} \sin\left(\sqrt{-\mu}\right) A + \sqrt{-\mu} \cos\left(\sqrt{-\mu}\right) B = 0 \tag{44}$$

which leads to

$$B=0, \qquad \sin\left(\sqrt{-\mu}\right) A=0. \tag{45}$$

We see that nonzero solution is only possible when $\sin\left(\sqrt{-\mu}\right)=0$ or equivalently

$$\mu=-n^2\pi^2. \tag{46}$$

In other words, these values belong to eigenvalues, and for each eigenvalue $\mu = -n^2\pi^2$, the corresponding eigenfunctions are

$$Y(y)=A\cos(n\pi y). \tag{47}$$

iii. Summarizing, we know that the eigenvalues are

$$0, \ -n^2\pi^2, \quad n=1,2,3,\ldots \tag{48}$$

with corresponding eigenfunctions

$$Y_0(y)=A, \ Y_n(y)=A\cos(n\pi y), \quad n=1,2,3,\ldots. \tag{49}$$

Since $-0^2\pi^2=0$ and $\cos(0\pi y)=1$, we can simply write

$$\mu_n=-n^2\pi^2, \qquad Y_n(y)=A\cos(n\pi y), \qquad n=0,1,2,3,\ldots \tag{50}$$

From solving the eigenvalue problem, we have concluded that the only values of μ which would allow nonzero Y are $\mu_n,\ n=0,1,2,3,\ldots$, and the corresponding solutions are constant multiples of $\cos(n\pi y)$. This constant multiple will be cancelled out in the final formula of the solution. Therefore we will take

$$\mu_n=-n^2\pi^2, \qquad Y_n(y)=\cos(n\pi y), \qquad n=0,1,2,3,\ldots \tag{51}$$

(It's perfectly OK to take other nonzero constants, for example $Y_0=3$, $Y_1=2.7\cos(\pi y)$, $Y_2=\sqrt{5}\cos(2\pi y)$, etc.)

Solving X.

Now that we have obtained all possible values of μ, we move on to solve X or Z. Compare: The problem for X involves one unknown constant λ; The problem for Z involves one unkonwn constant λ and is missing one boundary condition. It is clear that we should solve X now.

$$X''+(\lambda+\mu)X=0, \qquad X'(0)=X'(1)=0; \tag{52}$$

The problem for X is similar to that of Y so we omit the details. For each n (thus fix μ_n) the λ values that allow nonzero solutions are those satisfying

$$\lambda+\mu_n=m^2\pi^2 \qquad m=0,1,2,3,\ldots \tag{53}$$

with corresponding solutions

$$X=A\cos(m\pi x). \tag{54}$$

Therefore the eigenvalues are

$$\lambda_{mn}=\left(m^2+n^2\right)\pi^2 \qquad m,n=0,1,2,\ldots \tag{55}$$

with corresponding eigenfunctions

$$X_{mn}=A\cos(m\pi x). \tag{56}$$

Summary. We have found out that the only values of λ,μ which allow non-zero solutions are

$$\lambda_{mn}=\left(m^2+n^2\right)\pi^2, \qquad \mu_n=-n^2\pi^2, \qquad m,n=0,1,2,3,\ldots \tag{57}$$

with corresponding nonzero solutions (constant multiple of)

$$X_{mn}(x)=\cos(m\pi x), \qquad Y_n(y)=\cos(n\pi y). \tag{58}$$

4. **Finishing the solution**.

What to do.

Finally we need to solve Z. To obtain the boundary condition at 0 for Z, recall that

$$u(x,y,z)=\sum_{m,n} U_{mn}(x,y,z)=\sum_{m,n} X_{mn}(x)\,Y_n(y)\,Z_{mn}(z). \tag{59}$$

The (so-far haven't been used) boundary condition

$$u_z(x,y,0)=\cos\pi x\cos\pi y \tag{60}$$

leads to

$$\cos\pi x\cos\pi y=\sum_{m,n} Z'_{mn}(0)\,X_{mn}(x)\,Y_n(y)=\sum_{m,n} Z'_{mn}(0)\cos(m\pi x)\cos(n\pi y). \tag{61}$$

Thus all we need to do is to expand $\cos\pi x\cos\pi y$ into $\cos(m\pi x)\cos(n\pi y)$.

How to expand.

We know that we can expand any one-variable function into Fourier-cosine series over (0, 1) (note that $l=1$ here):

$$f(y)=\sum_{n=0}^{\infty} f_n\cos(n\pi y) \tag{62}$$

with

$$f_n=2\int_0^1 f(y)\cos(n\pi y)\,\mathrm{d}y. \tag{63}$$

The formula can be derived by assuming the correctness of the expansion and computing

$$\int_0^1 f(y)\cos(n\pi y)\,\mathrm{d}y=\sum_{m=0}^{\infty} f_m\int_0^1 \cos(m\pi y)\cos(n\pi y)\,\mathrm{d}y.$$

Now if instead of $f(y)$ we have $f(x,\,y)$, we can still treat x as a parameter and do the same thing:

$$f(x,y)=\sum_{n=0}^{\infty} f_n(x)\cos(n\pi y), \qquad f_n(x)=2\int_0^1 f(x,y)\cos(n\pi y)\,\mathrm{d}y. \tag{64}$$

We can then expand each $f_n(x)$:

$$f_n(x)=\sum_{m=0}^{\infty} f_{mn}\cos(m\pi x), \qquad f_{mn}=2\int_0^1 f_n(x)\cos(m\pi x)\,\mathrm{d}x. \tag{65}$$

Now substituting $f_n(x)$ by its expansion, we have

$$\begin{aligned} f(x,y) &= \sum_{n=0}^{\infty} f_n(x)\cos(n\pi y) \\ &= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} f_{mn}\cos(m\pi x)\right]\cos(n\pi y) \\ &= \sum_{m,n} f_{mn}\cos(m\pi x)\cos(n\pi y). \end{aligned} \tag{66}$$

And

$$f_{mn}=2\int_0^1 f_n(x)\cos(m\pi x)\,\mathrm{d}x=4\int_0^1\int_0^1 f(x,y)\cos(m\pi x)\cos(n\pi y)\,\mathrm{d}x\,\mathrm{d}y. \tag{67}$$

From this we conclude that the expansion of $\cos\pi x\cos\pi y$ is

$$\sum_{m,n} f_{mn}\cos(m\pi x)\cos(n\pi y) \tag{68}$$

with

$$f_{m n} = 4 \int_0^1 \int_0^1 \cos(\pi x) \cos(\pi y) \cos(m \pi x) \cos(n \pi y) \, \mathrm{d}x \, \mathrm{d}y. \tag{69}$$

As

$$2 \int_0^1 \cos(\pi x) \cos(m \pi x) \, \mathrm{d}x = \begin{cases} 1 & m = 1 \\ 0 & m \neq 1 \end{cases} \tag{70}$$

we see that

$$f_{m n} = \begin{cases} 1 & m = n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{71}$$

Now recalling

$$\cos \pi x \cos \pi y = \sum_{m,n} Z'_{m n}(0) \, X_{m n}(x) \, Y_n(y) = \sum_{m,n} Z'_{m n}(0) \cos(m \pi x) \cos(n \pi y) \tag{72}$$

we see that

$$Z'_{m n}(0) = \begin{cases} 1 & m = n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{73}$$

Solving $\boldsymbol{Z}$. Now we can solve Z. For each m, n we have

$$Z''_{m n} - \left(m^2 + n^2\right) \pi^2 \, Z_{m n} = 0, \qquad Z'_{m n}(1) = 0, \quad Z'_{m n}(0) = \begin{cases} 1 & m = n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{74}$$

First we write down the general solution.

There are two cases. When $m = n = 0$, we have

$$Z_{00} = A + B z, \tag{75}$$

otherwise

$$Z_{m n}(z) = A \, e^{\sqrt{m^2 + n^2} \pi z} + B \, e^{-\sqrt{m^2 + n^2} \pi z}. \tag{76}$$

Imposing the boundary conditions we conclude that

$$Z_{00} = A, \tag{77}$$

$$Z_{m n}(z) = 0 \qquad m \neq 1 \text{ or } n \neq 1. \tag{78}$$

For Z_{11}, we have

$$Z_{11}(z) = A \, e^{\sqrt{2} \pi z} + B \, e^{-\sqrt{2} \pi z}. \tag{79}$$

Boundary conditions lead to

$$Z'_{11}(0) = 1 \implies \sqrt{2} \, \pi \, [A - B] = 1 \tag{80}$$

$$Z'_{11}(0) = 0 \implies \sqrt{2} \, \pi \left[A \, e^{\sqrt{2} \pi} - B \, e^{-\sqrt{2} \pi} \right] = 0 \implies A \, e^{\sqrt{2} \pi} - B \, e^{-\sqrt{2} \pi} = 0. \tag{81}$$

Solving this system we obtain

$$A = \frac{1}{\sqrt{2} \, \pi} \frac{1}{1 - e^{2 \sqrt{2} \pi}}, \qquad B = \frac{1}{\sqrt{2} \, \pi} \frac{1}{e^{-2 \sqrt{2} \pi} - 1}. \tag{82}$$

Thus

$$Z_{11}(z) = \frac{1}{\sqrt{2} \, \pi} \frac{e^{\sqrt{2} \pi (z - 1)} + e^{-\sqrt{2} \pi (z - 1)}}{e^{-\sqrt{2} \pi} - e^{\sqrt{2} \pi}}. \tag{83}$$

Putting everything together, we have

$$u(x, y, z) = \frac{1}{\sqrt{2} \, \pi} \frac{e^{\sqrt{2} \pi (z - 1)} + e^{-\sqrt{2} \pi (z - 1)}}{e^{-\sqrt{2} \pi} - e^{\sqrt{2} \pi}} \cos(\pi x) \cos(\pi y) + C. \tag{84}$$

Please take a look at the problem again to see why the arbitrary constant C appear.

3. Other issues related to separation of variables.

3.1. Coordinate system.

From the above example we see that for the variables to be successfully separated, the domain of the problem has to be of the form $a_1 < x < a_2$; $b_1 < y < b_2$; $c_1 < z < c_2$; ... since otherwise, the interval of the equations for $X, Y, Z, \ldots$ would involve other variables.

Therefore, when the domain is not of the above form, to solve the problem using separation of variables, we need to first choose a new set of coordinates so that the domain becomes the above "rectangular" form in these new variables. For example

- The cylindrical (in Cartesian coordinates) domain $x^2 + y^2 < a^2$, $0 < z < l$ becomes the rectangular domain
$$0 \leqslant r < a, \quad 0 \leqslant \theta < 2\pi, \quad 0 < z < l \tag{85}$$
in cylindrical coordinates.
- The disc $x^2 + y^2 < a^2$ becomes the rectangle
$$0 \leqslant r < a, \quad 0 \leqslant \theta < 2\pi \tag{86}$$
in polar coordinates.
- The sphere $x^2 + y^2 + z^2 < a^2$ becomes the rectangular domain
$$0 \leqslant r < a, \quad 0 \leqslant \theta < 2\pi, \quad 0 \leqslant \varphi < \pi \tag{87}$$
in the spherical coordinates.

3.2. The role of the Sturm-Liouville theory.

Except for simplest problems, the eigenvalue problems involved in separation of variables leads to eigenfunctions which are not of the form $\cos\left(\frac{n\pi}{l} x\right)$ and $\sin\left(\frac{n\pi}{l} x\right)$. Thus in general, the theory of Fourier series cannot guarantee the validity of expansions of the solution like the following
$$u(x, y, z) = \sum_{m,n} X_{mn}(x)\, Y_n(y)\, Z_{mn}(z). \tag{88}$$
Instead, the validity of such expansions should be checked using the Sturm-Liouville theory.

3.3. Correct expansion.

Another important contribution of the Sturm-Liouville theory is making it possible to expand a function against a set of eigenfunctions which are not $\cos\left(\frac{n\pi}{l} x\right)$ and $\sin\left(\frac{n\pi}{l} x\right)$.

For example, when solving
$$u_{xx} + u_{yy} + u_{zz} = 0 \tag{89}$$
over the cylindrical domain $x^2 + y^2 < a^2$, $0 < z < l$, we first change variables to r, θ, z, and then the Θ equation yields eigenvalues n^2 and eigenfunctions $\cos(n\theta), \sin(n\theta)$. The R equation then becomes
$$r^2 R'' + r R' + \left(\lambda r^2 - n^2\right) R = 0 \tag{90}$$
which leads to
$$\lambda_{mn} = \alpha_{mn}/a, \qquad R_{mn} = J_n\!\left(\frac{\alpha_{mn}}{a} r\right). \tag{91}$$
The Sturm-Liouville theory tells us that
$$\int_0^a R_{mn}(r)\, R_{kn}(r)\, r\, \mathrm{d}r = 0 \tag{92}$$
which means if
$$f(r) = \sum_{m=1}^{\infty} f_m\, R_{mn}(r) \tag{93}$$
then
$$f_m = \frac{\int_0^a f(r)\, R_{mn}(r)\, r\, \mathrm{d}r}{\int_0^a R_{mn}(r)^2\, r\, \mathrm{d}r}. \tag{94}$$
This allows us to expand the boundary values into the eigenfunctions and finally solve the problems.

4. Green's function.

4.1. Green's function for ODEs.

Computing the Green's functions.

The Green's function $G(x;\xi)$ for an ODE

$$L y = f \tag{95}$$

with boundary conditions (left and right)

$$(B_L y)(a) = A, \qquad (B_R y)(a) = B, \tag{96}$$

is obtained through the following steps.

1. Multiply by appropriate $h(x)$ and write the LHS of the equation into the S-L form

$$h\, L y = (p\, y')' + q\, y \tag{97}$$

2. Writing down the general solution of the homogeneous equation

$$L y = 0. \tag{98}$$

3. Imposing the left boundary condition, we get solutions which are constant multiples of one another. Pick any one of them, call it y_L. Imposing the right boundary condition and obtain y_R.

4. Compute the constant

$$C = p(\xi) \left[y_L(\xi)\, y_R'(\xi) - y_R(\xi)\, y_L'(\xi) \right] \tag{99}$$

5. The Green's function is given by

$$G(x;\xi) = \begin{cases} \dfrac{y_R(\xi)}{C}\, y_L(x) & a < x < \xi \\ \dfrac{y_L(\xi)}{C}\, y_R(x) & \xi < x < b \end{cases}. \tag{100}$$

   To remember this formula, just keep in mind that since y_L satisfies the left boundary condition and y_R the right, the only way to combine them to obtain G is to let G be a multiple of y_L on the left half $x < \xi$ and a multiple of y_R on the right half $x > \xi$.

Remark 2. From the above discussion we clearly see that $G(x;\,\xi)$ depends on the operators L, B_L, B_R but is independent of the RHSs f, A, B.

Remark 3. It is also clear that the Green's function corresponding to (L, B_L, B_R) and $(f\, L, B_L, B_R)$ are the same for any f.

Solving ODEs using Green's functions.

Suppose we have a problem

$$L y = f(x), \qquad (B_L y)(a) = A, \quad (B_R y)(b) = B. \tag{101}$$

Here L is a second order differential operator, $B_L y$, $B_R y$ are combinations of y and y'. To solve this problem using Green's functions we need to follow the following steps.

1. Solve the problem

$$L y = 0, \quad (B_L y)(a) = A, \quad (B_R y)(b) = B. \tag{102}$$

   Let the solution be denoted by $w(x)$. Denote $u = y - w$, then the equations for u are

$$L u = L(y - w) = L y - L w = L y = f, \qquad (B_L u)(a) = 0, \quad (B_R y)(b) = 0. \tag{103}$$

2. Multiply by appropriate $h(x)$ and write the LHS of the equation into the form

$$h\, L y = (p\, y')' + q\, y = h\, f. \tag{104}$$

3. Let $G(x;\xi)$ be the Green's function corresponding to (L, B_L, B_R). Then

$$u(x) = \int_a^b G(x;\xi)\, h(\xi)\, f(\xi). \tag{105}$$

4. Finally we have

$$y = u + w. \tag{106}$$

4.2. Green's function for PDEs.

We only discuss the problem

$$u_{xx} + u_{yy} = f(x, y) \qquad (x, y) \in D \tag{107}$$

with certain boundary conditions.

The Green's function $G(x,\ y;\ \xi,\ \eta)$ satisfies the homogeneous counterparts of the equation as well as the boundary conditions. For example,

$$u_{xx} + u_{yy} = f(x, y) \qquad x > 0, y > 0, \qquad u = g_1 \text{ along } x = 0, \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial n} = g_2 \text{ along } y = 0 \tag{108}$$

Then the corresponding Green's function satisfies

$$G_{\xi\xi} + G_{\eta\eta} = 0, \qquad G = 0 \text{ along } \xi = 0,\ \frac{\partial G}{\partial \eta} = -\frac{\partial G}{\partial n} = 0 \text{ along } \eta = 0. \tag{109}$$

The procedure of finding the Green's function consists of the following steps.

1. The fundamental solution:

$$\Gamma(x - \xi, y - \eta) = \frac{1}{2\pi} \ln\left(\sqrt{(x - \xi)^2 + (y - \eta)^2}\right). \tag{110}$$

2. There are several ways to construct G from Γ.

   i. The mathematical way.

   Consider possible combinations of Γ of the form

$$G = \Gamma + \sum a_i\, \Gamma(b_i\, (X_i - \xi), b_i\, (Y_i - \eta)) \tag{111}$$

   and select appropriate a_i, b_i, X_i, Y_i so that the conditions are satisfied.

   ii. The "physical" way.

   The physical meaning of $\Gamma(x - \xi,\ y - \eta)$ (as a function of $(\xi,\ \eta)$) is the potential generated by a unit charge at (x, y). To obtain G, we need to put other charges at places outside $\xi > 0, \eta > 0$ to correctly cancel Γ along the boundary.

   For example, if we require $G = 0$ along $\eta = 0$, then it is clear that we should put a charge of the opposite sign at the point $(x, -y)$. In other words, adding

$$-\Gamma(x - \xi, -y - \eta) \tag{112}$$

   to Γ makes it 0 along $\eta = 0$.

   iii. The "smart" way.

   Say we would like $G = 0$ along $\eta = 0$. Setting $\eta = 0$ in Γ we see that its value is $\Gamma(x - \xi,\ y)$. The slightest change of Γ which can also give us this value along $\eta = 0$ is $\Gamma(x - \xi, y + \eta)$. As $\Gamma(x - \xi, y + \eta) = \Gamma(x - \xi, -y - \eta)$, this function satisfies

$$\Gamma(x - \xi, -y - \eta)_{\xi\xi} + \Gamma(x - \xi, -y - \eta)_{\eta\eta} = 0 \tag{113}$$

   for $\xi > 0, \eta > 0$.

Solution formulas for Dirichlet ($u = g$ on B) and Neumann ($\frac{\partial u}{\partial n} = g$ on B) problems:

$$u(x, y) = \int_B \frac{\partial \Gamma}{\partial n}\, g\, \mathrm{d}s + \int\int_D \Gamma\, (x - \xi, y - \eta)\, f(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \qquad \text{(Dirichlet)} \tag{114}$$

$$u(x, y) = -\int_B \Gamma\, g\, \mathrm{d}s + \int\int_D \Gamma\, (x - \xi, y - \eta)\, f(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \qquad \text{(Neumann)} \tag{115}$$

5. Transformation method.

5.1. The Fourier transform.

The Fourier transform of a function $f(x)$ is

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,\mathrm{d}x. \tag{116}$$

The inverse transform is

$$\mathcal{F}^{-1}(u)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} u(\xi)\,\mathrm{d}\xi. \tag{117}$$

5.2. When should it be used.

For problems whose domain involves the full range of one or more variables. For example, if the domain is $\{-\infty < x < \infty, y > 0\}$, Fourier transform (w.r.t. x) should be used; On the contrary, if the domain is $\{a < x < b, y > 0\}$, Fourier series (w.r.t. x) should be used.

5.3. Apply Fourier transform to solve PDEs.

The following steps should be following when solving a PDE

$$L u = f, \qquad \text{Initial conditions} \tag{118}$$

1. Take the Fourier transform of the equation as well as the initial conditions.
2. Solve the resulting equation to obtain the Fourier transform of the solution.
3. Take the inverse Fourier transform to obtain the formula for u.