1. Fourier transforms.

1.1. Definition.

As we have seen (at the end of the lectures on Fourier series), when the period of the function becomes larger and larger, the relation between the Fourie coefficients and the function approaches an integral transform/inverse transform. We call this transform the Fourier transform.

Definition 1. (Fourier transform) *The Fourier transform of a function $f(x)$ is*

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,\mathrm{d}x. \tag{1}$$

The inverse transform is

$$\mathcal{F}^{-1}(u)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} u(\xi)\,\mathrm{d}\xi. \tag{2}$$

Note that we use ξ instead of k here, as in general k denotes discrete numbers.

Example 2. (§12.18 1 a)) The most important Fourier transform formula in practice is

$$\mathcal{F}\left(e^{-ax^2}\right)(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}. \tag{3}$$

In particular,

$$\mathcal{F}\left(e^{-x^2/2}\right)(\xi) = e^{-\xi^2/2}. \tag{4}$$

(Formal) Proof. We compute

$$\begin{aligned}
\mathcal{F}\left(e^{-ax^2}\right)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} e^{-ax^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\xi}{2a}\right)^2 - \frac{\xi^2}{4a}}\,\mathrm{d}x \\
&= \frac{e^{-\frac{\xi^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\xi}{2a}\right)^2}\,\mathrm{d}x \\
&= e^{-\frac{\xi^2}{4a}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty+\frac{i\xi}{2a}}^{\infty+\frac{i\xi}{2a}} e^{-ay^2}\,\mathrm{d}y \right) \\
&= \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty+\frac{i\xi}{2\sqrt{a}}}^{\infty+\frac{i\xi}{2\sqrt{a}}} e^{-y^2}\,\mathrm{d}y \right) \\
&= \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}.
\end{aligned} \tag{5}$$

Here we have used the fact that

$$\int_{-\infty+\frac{i\xi}{2\sqrt{a}}}^{\infty+\frac{i\xi}{2\sqrt{a}}} e^{-y^2}\,\mathrm{d}y = \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y = \sqrt{\pi}. \tag{6}$$

The proof for the first equality can be found in any Complex Analysis textbook. The proof for the second equality is as follows.

We introduce a new variable x. Then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y &= \left[\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\right]^{1/2}\\ &= \left(\iint e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\\ &= \left(\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta\right)^{1/2}\\ &= \left(\pi\int_0^{\infty} e^{-z}\,\mathrm{d}z\right)^{1/2}\\ &= \sqrt{\pi}. \end{aligned} \tag{7}$$

Example 3. (§12.18 2) Find the Fourier transform of the gate function

$$f_a(x) = \begin{cases} 1 & |x| < a \quad a \text{ is a positive constant} \\ 0 & |x| > a \end{cases}. \tag{8}$$

Solution. We compute

$$\begin{aligned}\mathcal{F}(f_a(x))(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} f_a(x)\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ix\xi}\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi}}\frac{i}{\xi}\int_{ia\xi}^{-ia\xi} e^{z}\,\mathrm{d}z\\ &= \frac{1}{\sqrt{2\pi}}\frac{i}{\xi}\left(e^{-ia\xi} - e^{ia\xi}\right)\\ &= \frac{1}{\sqrt{2\pi}}\frac{i}{\xi}(-2i)\sin(a\xi)\\ &= \sqrt{\frac{2}{\pi}}\frac{\sin(a\xi)}{\xi}. \end{aligned} \tag{9}$$

1.2. Properties.

- The Fourier transform is linear, that is

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g). \tag{10}$$

- We have

$$\mathcal{F}(f(\cdot - c))(\xi) = e^{-ixc}\,\mathcal{F}(f)(\xi). \tag{11}$$

- We have

$$\mathcal{F}(f(c\,\cdot))(\xi) = \frac{1}{|c|}\mathcal{F}(f)(\xi/c). \tag{12}$$

- We have

$$\mathcal{F}(f')(\xi) = i\,\xi\,\mathcal{F}(f)(\xi) \tag{13}$$

  or more generally

$$\mathcal{F}\left(f^{(n)}\right)(\xi) = (i\,\xi)^n\,\mathcal{F}(f)(\xi). \tag{14}$$

- Parseval's formula

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |\mathcal{F}(f)(\xi)|^2\,\mathrm{d}\xi. \tag{15}$$

1.3. Convolutions.

The convolution of two functions f, g are a new function defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,\mathrm{d}y \tag{16}$$

denoted $f * g$.

The convolution operation enjoys the following nice properties, which make it very useful in PDE theory.

- Basic properties

$$\begin{aligned} f*g &= g*f && (17)\\ f*(g*h) &= (f*g)*h && (18)\\ f*(a\,g+b\,h) &= a\,(f*g)+b\,(f*h) && (19)\end{aligned}$$

  where a, b are arbitrary constants.

- Fourier transform of convolutions.

$$\mathcal{F}(f*g)(\xi)=[\mathcal{F}(f)(\xi)]\,[\mathcal{F}(g)(\xi)]. \tag{20}$$

- Differentiation of convolutions.

$$(f*g)^{(n)}=f^{(k)}*g^{(n-k)}. \tag{21}$$

1.4. Applications of Fourier transforms to PDEs.

Example 4. (§12.18 11) Determine the solution of the initial-value problem

$$u_{tt}=c^2\,u_{xx}\qquad -\infty<x<\infty,\quad t>0 \tag{22}$$

$$u(x,0)=f(x),\qquad u_t(x,0)=g(x). \tag{23}$$

Solution. We take Fourier transform in the x variable. Let $U(\xi,t)$ be the result. Then we have

$$u_{tt}\longrightarrow U_{tt},\qquad u_{xx}\longrightarrow -\xi^2\,U \tag{24}$$

and the equation becomes

$$U_{tt}+c^2\,\xi^2\,U=0,\qquad U(\xi,0)=\mathcal{F}(f),\qquad U_t(\xi,0)=\mathcal{F}(g). \tag{25}$$

The general solution is

$$U(\xi,t)=A(\xi)\cos(c\,\xi\,t)+B(\xi)\sin(c\,\xi\,t). \tag{26}$$

To fix A, B we use the initial values:

$$\mathcal{F}(f)(\xi)=U(\xi,0)=A(\xi), \tag{27}$$

$$\mathcal{F}(g)(\xi)=U_t(\xi,0)=c\,\xi\,B(\xi). \tag{28}$$

Therefore

$$U(\xi,t)=\mathcal{F}(f)(\xi)\cos(c\,\xi\,t)+\mathcal{F}(g)(\xi)\,\frac{\sin(c\,\xi\,t)}{c\,\xi}. \tag{29}$$

Taking the inverse transform we obtain

$$u(x,t)=f*\mathcal{F}^{-1}(\cos(c\,\xi\,t))+g*\mathcal{F}^{-1}\left(\frac{\sin(c\,\xi\,t)}{c\,\xi}\right). \tag{30}$$

To obtain the formula in variables x, t we need to compute

$$\mathcal{F}^{-1}(\cos(c\,\xi\,t))\ \text{ and }\ \mathcal{F}^{-1}\left(\frac{\sin(c\,\xi\,t)}{c\,\xi}\right). \tag{31}$$

It is easy to see that direct integration using the inverse transform formula does not quite work. We need to compute them indirectly.

As

$$\cos(c\,\xi\,t)=\frac{1}{2}\left(e^{ic\xi t}+e^{-ic\xi t}\right), \tag{32}$$

we have

$$\mathcal{F}^{-1}(\cos(c\,\xi\,t))(x)=\frac{1}{2}\left[\mathcal{F}^{-1}\left(e^{ic\xi t}\right)+\mathcal{F}^{-1}\left(e^{-ic\xi t}\right)\right]=\frac{1}{2}\left[\mathcal{F}^{-1}(1)(x+c\,t)+\mathcal{F}^{-1}(1)(x-c\,t)\right]. \tag{33}$$

To find out $\mathcal{F}^{-1}(1)$, we use the property of convolution. We have

$$\mathcal{F}^{-1}(1) * f = \mathcal{F}^{-1}\big(\mathcal{F}\big(\mathcal{F}^{-1}(1)\big)\,\mathcal{F}(f)\big) = \mathcal{F}^{-1}(\mathcal{F}(f)) = f. \tag{34}$$

Therefore

$$\frac{1}{\sqrt{2\,\pi}} \int_{\infty}^{\infty} \mathcal{F}^{-1}(1)(x-y)\, f(y)\,\mathrm{d}y = f(x) \tag{35}$$

for all x. This tells us that

$$\mathcal{F}^{-1}(1)(x) = \sqrt{2\,\pi}\,\delta(x). \tag{36}$$

As a consequence

$$\mathcal{F}^{-1}(\cos(c\,\xi\,t)) = \sqrt{\frac{\pi}{2}}\ [\delta(x+c\,t) + \delta(x-c\,t)]. \tag{37}$$

For the second term, recall that we have shown that

$$\mathcal{F}(f_a)(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(a\,\xi)}{\xi} \tag{38}$$

where $f_a = 1$ for $|x| < a$ and 0 outside. As a consequence

$$\mathcal{F}^{-1}\bigg(\frac{\sin(c\,\xi\,t)}{c\,\xi}\bigg) = \frac{1}{c}\,\mathcal{F}^{-1}\bigg(\frac{\sin(c\,\xi\,t)}{\xi}\bigg) = \sqrt{\frac{\pi}{2}}\,\frac{1}{c}\,f_{ct}. \tag{39}$$

Finally we compute

$$\begin{aligned}
u(x,t) &= f * \mathcal{F}^{-1}(\cos(c\,\xi\,t)) + g * \mathcal{F}^{-1}\bigg(\frac{\sin(c\,\xi\,t)}{c\,\xi}\bigg) \\
&= \sqrt{\frac{\pi}{2}}\,[f * \delta(x+c\,t) + f * \delta(x-c\,t)] \\
&\quad + \sqrt{\frac{\pi}{2}}\,\frac{1}{c}\,g * f_{ct} \\
&= \sqrt{\frac{\pi}{2}}\bigg[\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} f(y)\,\delta(x+c\,t-y)\,\mathrm{d}y + \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} f(y)\,\delta(x-c\,t-y)\,\mathrm{d}y\bigg] \\
&\quad + \sqrt{\frac{\pi}{2}}\,\frac{1}{c}\,\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} g(y)\,f_{ct}(x-y)\,\mathrm{d}y \\
&= \frac{1}{2}\,[f(x+c\,t) + f(x-c\,t)] + \frac{1}{2\,c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y.
\end{aligned} \tag{40}$$

This is just the d'Alembert formula.

Example 5. (§12.18 13) Solve

$$u_{t\,t} + c^2\,u_{x\,x\,x\,x} = 0, \qquad -\infty < x < \infty, \qquad t > 0 \tag{41}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = 0. \tag{42}$$

Solution. We take the Fourier transform (in x) of the equation and the initial values, and obtain

$$U_{t\,t} + c^2\,\xi^4\,U = 0, \qquad U(\xi,0) = \mathcal{F}(f)(\xi), \qquad U_t(\xi,0) = 0. \tag{43}$$

The general solution is

$$U(\xi,t) = A(\xi)\cos\big(c\,\xi^2\,t\big) + B(\xi)\sin\big(c\,\xi^2\,t\big). \tag{44}$$

Using the initial values we reach

$$U(\xi,t) = \mathcal{F}(f)(\xi)\cos\big(c\,\xi^2\,t\big). \tag{45}$$

Therefore

$$u(x,t) = \mathcal{F}^{-1}\big[\mathcal{F}(f)(\xi)\cos\big(c\,\xi^2\,t\big)\big] = \mathcal{F}^{-1}\big(\cos\big(c\,\xi^2\,t\big)\big) * f. \tag{46}$$