WEEKS 07 – 08: SEPARATION OF VARIABLES

In the past few weeks we have explored the possibility of solving first and second order PDEs by transforming them into simpler forms (method of characteristics). Unfortunately, this process often does not help much. If we start with an arbitrary second order PDE, and reduce it to canonical form, most likely we still do not know how to find the general solutions. In the following few weeks, we will introduce a method that are much more powerful at finding solutions than the method of characteristics. The main idea is separation of variables.

1. The vibrating string problem revisited.

We consider the system

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < l, \; t > 0 && (1)\\
u(x,0) &= f(x), \quad 0 \leqslant x \leqslant l, && (2)\\
u_t(x,0) &= g(x), \quad 0 \leqslant x \leqslant l, && (3)\\
u(0,t) &= 0, \qquad t \geqslant 0, && (4)\\
u(l,t) &= 0, \qquad t \geqslant 0. && (5)
\end{aligned}
$$

In the past week we have tried to solve this system using the general solution formulas of the wave equations. We have seen that it is not possible to write a clean formula for the solution and therefore very hard to extract much information from it. Now we try another approach.

Instead of trying to get the general solution, we ask, what is the simplest nontrivial solution of the wave equation (without considering the initial and boundary conditions)? One possibility is the special form

$$u(x,t) = X(x)\,T(t). \tag{6}$$

We plug this into the equation to see when we will be so lucky to have such solutions. We obtain

$$X\,T'' = c^2 X''\,T \tag{7}$$

which leads to

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}. \tag{8}$$

This seems to be a complicated equation until we realized that X''/X is a function of x only and T''/T is a function of t only. The equality can hold only if both X''/X and T''/T are constants.[1]

From this we see that $u(x,\,t) = X(x)\;T(t)$ is a solution to the wave equation if there is a constant λ such that

$$
\begin{aligned}
X'' - \lambda X &= 0, && (9)\\
T'' - \lambda c^2 T &= 0. && (10)
\end{aligned}
$$

From ODE theory we know that the solutions are

$$X(x) = A\,e^{\lambda^{1/2}x} + B\,e^{-\lambda^{1/2}x}, \qquad T(t) = A\,e^{\lambda^{1/2}ct} + B\,e^{-\lambda^{1/2}ct}. \tag{11}$$

To fix the arbitrary constants, we now consider the boundary conditions $u(0,\,t) = u(l,\,t) = 0$ (but still neglect the initial conditions), which gives

$$X(0) = X(l) = 0. \tag{12}$$

Discussing the cases $\lambda > 0, = 0, < 0$ (see pp.236 – 237 of the textbook) we see that this can be satisfied by nonzero X only if

$$A = 0, \text{ and } \lambda = -\left(\frac{n\,\pi}{l}\right)^2. \tag{13}$$

Therefore

$$X(x) = X_n(x) \text{ for some } n \in \mathbb{N}, \tag{14}$$

1. To see this, differentiate the equation by $\frac{\partial}{\partial x}$, we have $\frac{\partial}{\partial x}\left(\frac{X''}{X}\right) = 0$ which gives $\frac{X''}{X} =$ constant because it is a function of x only.

where

$$X_n(x) = B_n \sin\left(\frac{n\,\pi}{l}\,x\right) \tag{15}$$

for some arbitrary constant B_n. It follows that

$$T(t) = T_n(t) = C_n \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + D_n \sin\left(\frac{n\,\pi\,c}{l}\,t\right) \tag{16}$$

for arbitrary constants C_n and D_n.

We have shown that all solutions of the form $X(t)\,T(t)$ are

$$\left(\alpha \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + \beta \sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right) \sin\left(\frac{n\,\pi}{l}\,x\right) \tag{17}$$

where α, β are constants.

Now it's time to take into account the initial conditions. If we take $t = 0$, we have

$$f(x) = u(x, 0) = \alpha \sin\left(\frac{n\,\pi}{l}\,x\right), \tag{18}$$

$$g(x) = u_t(x, 0) = \beta\,\frac{n\,\pi\,c}{l} \sin\left(\frac{n\,\pi}{l}\,x\right). \tag{19}$$

which are clearly not true for most f and g!

Does this mean the method fails? Not so fast. Recall that for linear equations, finite or even infinite sums of solutions are still solutions (certain conditions apply). Therefore, if we can find finitely many constants α_n, β_n: $n = 1, \ldots, m$ or infinitely many constants α_n', β_n' such that

$$f(x) = \sum_1^m \alpha_n \sin\left(\frac{n\,\pi}{l}\,x\right), \qquad g(x) = \sum_1^m \beta_n\,\frac{n\,\pi\,c}{l} \sin\left(\frac{n\,\pi}{l}\,x\right), \tag{20}$$

or

$$f(x) = \sum_1^\infty \alpha_n \sin\left(\frac{n\,\pi}{l}\,x\right), \qquad g(x) = \sum_1^\infty \beta_n\,\frac{n\,\pi\,c}{l} \sin\left(\frac{n\,\pi}{l}\,x\right), \tag{21}$$

then the solution u can be written as either

$$u(x, t) = \sum_1^m \left(\alpha_n \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + \beta_n \sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right) \sin\left(\frac{n\,\pi}{l}\,x\right) \tag{22}$$

or

$$u(x, t) = \sum_1^\infty \left(\alpha_n \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + \beta_n \sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right) \sin\left(\frac{n\,\pi}{l}\,x\right). \tag{23}$$

It turns out that finite sum representation is in general not possible, we have to rely on the infinite sums. To indeed carry this out, we need to answer the following questions:

1. Can we represent arbitrary f, g?
2. If we can, how to compute the coefficients α_n, β_n?
3. Does the infinite sum represent a function? That is, does the sum converge?
4. If the infinite sum converges to a function, does this function solve the equation and satisfy the initial and boundary conditions?

The answers to these questions form the basics of the theory of Fourier series. We will study Fourier series in the following one to two lectures before returning to the method of separation of variables.

Remark 1. What is the relation between our new formula

$$u(x, t) = \sum_1^\infty \left(\alpha_n \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + \beta_n \sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right) \sin\left(\frac{n\,\pi}{l}\,x\right) \tag{24}$$

and our previous one using general solutions? Using basic trignometric formulas:

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \qquad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \tag{25}$$

we easily obtain

$$u(x,t)=\frac{1}{2}\sum_{1}^{\infty}\left\{\alpha_n\left[\sin\left(\frac{n\,\pi}{l}\,(x+c\,t)\right)-\sin\left(\frac{n\,\pi}{l}\,(x-c\,t)\right)\right]+\beta_n\left[\cos\left(\frac{n\,\pi}{l}\,(x-c\,t)\right)-\cos\left(\frac{n\,\pi}{l}\,(x+c\,t)\right)\right]\right\}$$

or equivalently

$$u(x,t)=\phi(x+c\,t)+\psi(x-c\,t) \tag{26}$$

with

$$\phi(x)=\frac{1}{2}\sum_{1}^{\infty}\alpha_n\sin\left(\frac{n\,\pi}{l}\,x\right)-\beta_n\cos\left(\frac{n\,\pi}{l}x\right), \tag{27}$$

$$\psi(x)=\frac{1}{2}\sum_{1}^{\infty}\beta_n\cos\left(\frac{n\,\pi}{l}x\right)-\alpha_n\sin\left(\frac{n\,\pi}{l}\,x\right). \tag{28}$$

We see that $\psi(x)=-\phi(-x)$ and $\phi(x)=-\psi(2\,l-x)$ as we know.

Remark 2. As we will soon see, oftentimes we cannot obtain a closed form formula for u and have to live with the infinite sum. Recalling our motivation – to do better than the method of characteristics – one may wonder what the difference is. The difference is the following. When we cannot solve the problem by method of characteristics, we are totally stuck; On the other hand, the coefficients in the infinite sum tell us much information of the solution.

2. Fourier Series.

Recall the issues we would like to settle.

1. Can we represent arbitrary f,g?
2. If we can, how to compute the coefficients α_n, β_n?
3. Does the infinite sum represent a function? That is, does the sum converge?
4. If the infinite sum converges to a function, does this function solve the equation and satisfy the initial and boundary conditions?

2.1. Representation of functions by Fourier series.

We first try to settle the first two questions. We study whether any function f can be represented in the form

$$f(x)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\,\pi}{l}\,x\right). \tag{29}$$

First we notice that, for any f representable in the above fashion, formally (that is neglecting any convergence issues related to infinite sums)

$$f(2\,k\,l)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\,\pi}{l}\,2\,k\,l\right)=\sum_{n=1}^{\infty}\alpha_n\sin(2\,k\,n\,\pi)=0, \tag{30}$$

$$\int_0^{2l}f(x)=\sum_{n=1}^{\infty}\alpha_n\int_0^{2l}\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}x=0, \tag{31}$$

$$f(x+2\,l)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\,\pi}{l}\,(x+2\,l)\right)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\,\pi}{l}\,x+2\,n\,\pi\right). \tag{32}$$

In other words, any function that is possible to be represented must satisfy

i. $f(2\,k\,l)=0$,

ii. $\int_0^{2l}f(x)\,\mathrm{d}x=0$,

iii. $f(x+2\,l)=f(x)$ that is f is a periodic function with period $2\,l$; Or equivalently, we can only represent functions defined over a interval of length $2\,l$.

As we would like to represent as many functions as possible, we would try to fix the above restrictions.

i. $f(2\,k\,l)=0$: The reason for this restriction is that $\sin\left(\frac{n\,\pi}{l}\,2\,k\,l\right)=0$, it is easily fixed by introducing $\cos\left(\frac{n\,\pi}{l}\,x\right)$ into the series.

ii. $\int_0^{2l} f(x)\,\mathrm{d}x=0$: Even with cos's introduced, we still have $\int_0^{2l} f(x)\,\mathrm{d}x=0$. Therefore we need to introduce one more term – a constant relating to $\int_0^{2l} f(x)\,\mathrm{d}x$ – into the series.

iii. $f(x+2\,l)=f(x)$: There is currently no way to fix this. We will mention a bit about this in a few lectures.

Now the representation becomes (we have changed the notations a bit to be consistent with the textbook)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\,\pi\,x}{l}\right)+b_k\sin\left(\frac{n\,\pi\,x}{l}\right)\right). \tag{33}$$

for f with period $2\,l$.

In the following we will show how to find the coefficients.

- a_0. We start with a_0. As we have seen, the introduction of a_0 is to allow us to represent function with nonzero mean. Therefore we integrate the representation over $(0,2\,l)$:

$$\int_0^{2l} f(x)\,\mathrm{d}x=\int_0^{2l}\frac{a_0}{2}\,\mathrm{d}x+\sum_{k=1}^{\infty}\left(a_k\int_0^{2l}\cos\left(\frac{k\,\pi\,x}{l}\right)\mathrm{d}x+b_k\int_0^{2l}\sin\left(\frac{k\,\pi\,x}{l}\right)\mathrm{d}x\right)=a_0\,l. \tag{34}$$

  Therefore

$$a_0=\frac{1}{l}\int_0^{2l} f(x)\,\mathrm{d}x. \tag{35}$$

- To obtain a_n and b_n, we notice

$$\int_0^{2l}\cos\left(\frac{n\,\pi\,x}{l}\right)\cos\left(\frac{m\,\pi\,x}{l}\right)\mathrm{d}x = \begin{cases} l & n=m\\ 0 & n\neq m\end{cases} \tag{36}$$

$$\int_0^{2l}\cos\left(\frac{n\,\pi\,x}{l}\right)\sin\left(\frac{m\,\pi\,x}{l}\right)\mathrm{d}x = 0 \tag{37}$$

$$\int_0^{2l}\sin\left(\frac{n\,\pi\,x}{l}\right)\sin\left(\frac{m\,\pi\,x}{l}\right)\mathrm{d}x = \begin{cases} l & n=m\\ 0 & n\neq m\end{cases} \tag{38}$$

  Therefore

$$a_n=\frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x; \qquad b_k=\frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x. \tag{39}$$

Remark 3. The interval $(0,2\,l)$ can be replaced by any interval of length $2\,l$.

From the above formulas, we clearly see that

$$f(x)\text{ is odd}\iff a_n=0\quad n=0,1,2,\ldots \tag{40}$$

$$f(x)\text{ is even}\iff b_n=0\quad n=1,2,\ldots \tag{41}$$

That is, when f is odd, only sin's are involved in the Fourier series and when f is even only cos's are involved. In these cases, we only need half of f to determine the coefficients:

$$f\text{ even:}\qquad a_n=\frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x,\quad b_n=0,\qquad n=0,1,2,\ldots \tag{42}$$

$$f\text{ odd:}\qquad b_n=\frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x,\quad a_n=0,\qquad n=1,2,\ldots \tag{43}$$

Example 4. (§6.14 1 a)) Find the Fourier series of the following function

$$f(x)=\begin{cases} x & -\pi<x<0 \\ h & 0<x<\pi \end{cases} \qquad h \text{ is a constant.} \tag{44}$$

Solution. As f is defined over $(-\pi,\pi)$, $l=\pi$. We compute

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x=\frac{1}{\pi}\left[\int_{-\pi}^{0} x\,\mathrm{d}x+\int_0^{\pi} h\,\mathrm{d}x\right]=\frac{1}{\pi}\left[-\frac{\pi^2}{2}+h\,\pi\right]=h-\frac{\pi}{2}; \tag{45}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos(n\,x)\,\mathrm{d}x+\int_0^{\pi} h\cos(n\,x)\,\mathrm{d}x\right] \\ &= \frac{1}{\pi}\left[\frac{1}{n}\int_{-\pi}^{0} x\,\mathrm{d}\sin(n\,x)+\frac{h}{n}\sin(n\,x)\,|_0^{\pi}\right] \\ &= \frac{1}{\pi}\left[\frac{1}{n}x\sin(n\,x)\,|_{-\pi}^{0}-\frac{1}{n}\int_{-\pi}^{0}\sin(n\,x)\,\mathrm{d}x+\frac{h}{n}\sin(n\,x)\,|_0^{\pi}\right] \\ &= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos(n\,x))\,|_{-\pi}^{0}\right] \\ &= \frac{1}{\pi\,n^2}\left[1-(-1)^n\right]. \end{aligned} \tag{46}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin(n\,x)\,\mathrm{d}x+\int_0^{\pi} h\sin(n\,x)\,\mathrm{d}x\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{n}\int_{-\pi}^{0} x\,\mathrm{d}\cos(n\,x)+h\int_0^{\pi}\sin(n\,x)\,\mathrm{d}x\right] \\ &= -\frac{1}{\pi\,n}\left[x\cos(n\,x)\,|_{-\pi}^{0}-\int_{-\pi}^{0}\cos(n\,x)\,\mathrm{d}x+h\cos(n\,x)\,|_0^{\pi}\right] \\ &= -\frac{1}{\pi\,n}\left[-(-\pi)(-1)^n-\frac{1}{n}\sin(n\,x)\,|_{-\pi}^{0}+h\left((-1)^n-1\right)\right] \\ &= -\frac{1}{\pi\,n}\left[\pi(-1)^n+h\left((-1)^n-1\right)\right] \\ &= \frac{1}{\pi\,n}\left[h-(h+\pi)(-1)^n\right]. \end{aligned} \tag{47}$$

Therefore the representation is

$$f(x)=\frac{h}{2}-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{1}{\pi\,n^2}\left[1-(-1)^n\right]\cos(n\,x)+\frac{1}{\pi\,n}\left[h-(h+\pi)(-1)^n\right]\sin(n\,x)\right). \tag{48}$$

Example 5. (§6.14 3 c)) Obtain the Fourier cosine series representation for the following functions:

$$f(x)=x^2, \qquad 0<x<\pi. \tag{49}$$

Solution. "Obtain the Fourier cosine series" effectively means extending f evenly and obtain its Fourier series. Or equivalently, use the formulas

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x, \quad b_n=0, \qquad n=0,1,2,\ldots \tag{50}$$

Thus $l=\pi$. We compute

$$a_0=\frac{2}{\pi}\int_0^{\pi} x^2\,\mathrm{d}x=\frac{2\,\pi^2}{3}. \tag{51}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2\cos(n\,x)\,\mathrm{d}x \\ &= \frac{2}{n\,\pi}\int_0^{\pi} x^2\,\mathrm{d}\sin(n\,x) \\ &= \frac{2}{n\,\pi}\left[x^2\sin(n\,x)\,|_0^{\pi} - \int_0^{\pi} 2\,x\sin(n\,x)\,\mathrm{d}x\right] \\ &= \frac{4}{n^2\,\pi}\int_0^{\pi} x\,\mathrm{d}\cos(n\,x) \\ &= \frac{4}{n^2\,\pi}\left[x\cos(n\,x)\,|_0^{\pi} - \int_0^{\pi}\cos(n\,x)\,\mathrm{d}x\right] \\ &= \frac{4\,(-1)^n}{n^2}. \end{aligned} \tag{52}$$

The Fourier cosine series representation is then

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\,(-1)^n}{n^2}\cos(n\,x). \tag{53}$$

Remark 6. Mathematically speaking, what we have done is the following. We have shown that, if a function $f(x)$ can be represented by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\,\pi\,x}{l}\right) + b_n\sin\left(\frac{n\,\pi\,x}{l}\right)\right). \tag{54}$$

Then the coefficients must be given by

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x, \qquad n = 0, 1, 2, \ldots \tag{55}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\,\pi\,x}{l}\right)\mathrm{d}x, \qquad n = 1, 2, \ldots \tag{56}$$

To establish a sound mathematical theory, we have to study whether the sequence

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\,\pi\,x}{l}\right) + b_n\sin\left(\frac{n\,\pi\,x}{l}\right)\right), \tag{57}$$

with coefficients given by the above formulas, converges to the function f.

2.2. Convergence and other issues.

We mention quickly theories relating the 3rd and 4th questions. In fact, due to limited time, we will only mention quickly all the questions that will be answered by the theory of Fourier series.

First we review what we have so far. We are concerned with solving the initial-boundary problem

$$\begin{aligned} u_{tt} - c^2\,u_{xx} &= 0, \qquad 0 < x < l,\ t > 0 && (58) \\ u(x,0) &= f(x), \quad 0 \leqslant x \leqslant l, && (59) \\ u_t(x,0) &= g(x), \quad 0 \leqslant x \leqslant l, && (60) \\ u(0,t) &= 0, \qquad t \geqslant 0, && (61) \\ u(l,t) &= 0, \qquad t \geqslant 0. && (62) \end{aligned}$$

We are able to construct the following infinite sum:

$$\sum_1^{\infty}\left(\alpha_n\cos\left(\frac{n\,\pi\,c}{l}\,t\right) + \beta_n\sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right)\sin\left(\frac{n\,\pi}{l}\,x\right) \tag{63}$$

and hope that this is our solution. Mathematically the following are required:

1. The sum gives a function. That is we can define

$$u(x,t)=\sum_{n=1}^{\infty}\left(\alpha_n\cos\left(\frac{n\,\pi\,c}{l}\,t\right)+\beta_n\sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right)\sin\left(\frac{n\,\pi}{l}\,x\right). \tag{64}$$

2. The equation is satisfies by this function. That is

   i. $\partial_{tt}u(x,t)$ and $\partial_{xx}u(x,t)$ are well-defined for $0<x<l, t>0$,

   ii. $u_{tt}-c^2\,u_{xx}=0$ for $0<x<l, t>0$.

   We note that if we can differentiate the sum termwise, then the equation will be satisfied.

3. Correct initial values are taken.

$$f(x) = \sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\,\pi}{l}\,x\right) \tag{65}$$

$$g(x) = \sum_{n=1}^{\infty}\beta_n\,\frac{n\,\pi\,c}{l}\sin\left(\frac{n\,\pi}{l}\,x\right) \tag{66}$$

   in appropriate senses, and furthermore

$$\lim_{t\searrow 0}u(x,t)=u(x,0),\qquad \lim_{t\searrow 0}u_t(x,t)=u_t(x,0). \tag{67}$$

   That is, the $t\searrow 0$ limit is the same as the result of setting $t=0$ in each term of the infinite sum.

4. Correct boundary values are taken.

$$\lim_{x\searrow 0}u(x,t)=u(0,t),\qquad \lim_{x\nearrow\infty}u(x,t)=u(l,t) \tag{68}$$

   where $u(0,t)$, $u(l,t)$ are the values of the infinite sum when we replace x by 0 and l in the infinite sum.

From the above we see that the following questions need to be answered for a Fourier series:

1. Given a function, is the corresponding Fourier series converging to this function?
2. Given an infinite sum of sin's and cos's, when is it convergent to a reasonable function?
3. If the series converges to a function, when is this function differentiable? Does the derivative coincide with the sum of termwise derivative of each term? That is if

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\,\pi}{l}\,x\right)+b_n\sin\left(\frac{n\,\pi}{l}\,x\right)\right) \tag{69}$$

   when do we have the existence of f', f'', etc. and furthermore when do we have

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p}f(x)=\sum_{n=1}^{\infty}\left(a_n\left(\frac{\mathrm{d}^p}{\mathrm{d}x^p}\cos\left(\frac{n\,\pi}{l}\,x\right)\right)+b_n\left(\frac{\mathrm{d}^p}{\mathrm{d}x^p}\sin\left(\frac{n\,\pi}{l}\,x\right)\right)\right). \tag{70}$$

4. Can we take termwise limits. That is

$$\lim_{x\to x_0}\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\,\pi\,x}{l}\right)+b_n\sin\left(\frac{n\,\pi\,x}{l}\right)\right)=?=\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\,\pi\,x_0}{l}\right)+b_n\sin\left(\frac{n\,\pi\,x_0}{l}\right)\right). \tag{71}$$

Of course in fact the above questions need to be answered for double Fourier series.

The answers to the above questions are highly involved. We just give the shortest (reads: crudest) answers here.

– Answer to 1.

  If f is piecewise continuous, then its Fourier series converges to f at all of its continuous points, and converges to $\frac{f(x+)+f(x-)}{2}$ at discontinuous points.

  In particular, if f is uniformly continuous, the convergence is uniform.

– Answer to 2 and 4.
  Yes as long as $\sum_1^\infty |a_n| + |b_n| < \infty$.
– Answer to 3.
  Yes as long as $\sum_1^\infty \left(\frac{n\pi}{l}\right)^p (|a_n| + |b_n|) < \infty$.

Remark 7. Finally we mention a few reasons why Fourier series is widely applied in modern mathematics and science.

a) Fourier series can be obtained for a wide class of functions.
b) The partial sum

$$f_N = \frac{a_0}{2} + \sum_1^N \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right) \tag{72}$$

is the best approximant of f in the following sense

$$f_N = \operatorname{argmin}_{g \in V_N} \int_0^{2l} (f-g)^2 \, \mathrm{d}x \tag{73}$$

where V_N is the space of all functions of the form

$$\frac{\alpha_0}{2} + \sum_1^N \left(\alpha_n \cos\left(\frac{n\pi x}{l}\right) + \beta_n \sin\left(\frac{n\pi x}{l}\right)\right). \tag{74}$$

Furthermore we have the Parseval's relation

$$\frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2) = \frac{1}{l} \int_0^{2l} f(x)^2 \, \mathrm{d}x. \tag{75}$$

c) Most importantly, the sin's and cos's are eigenfunctions of derivative operators. In particular, differentiation of one Fourier series results in another Fourier series. This is why Fourier series is ubiquitous in modern theory of PDEs.

2.3. Complex Fourier series.

Recalling

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2}, \tag{76}$$

we have

$$\frac{a_0}{2} + \sum_1^\infty \left(a_n \cos\left(\frac{n\pi}{l} x\right) + b_n \sin\left(\frac{n\pi}{l} x\right)\right) = c_0 + \sum_{k=1}^\infty \left(c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l}\right) = \sum_{-\infty}^\infty c_n e^{in\pi x/l}. \tag{77}$$

where

$$c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - i\, b_n}{2}, \qquad c_{-n} = \frac{a_n + i\, b_n}{2}. \tag{78}$$

This series is called the complex Fourier series.

Note that, instead of computing c_k from a_k and b_k, we can obtain the coefficients directly:

$$c_0 = \frac{1}{2l} \int_0^{2l} f(x) \, \mathrm{d}x, \tag{79}$$

$$c_n = \frac{1}{2l} \int_0^{2l} f(x)\, e^{-in\pi x/l} \, \mathrm{d}x, \qquad n = \pm 1, \pm 2, \ldots \tag{80}$$

Remark 8. We emphasize again that the interval $(0, 2l)$ can be replaced by any interval of length $2l$.

In this case the Parseval's relation takes a more aesthetically satisfactory form:

$$\sum_{-\infty}^\infty |c_n|^2 = \frac{1}{2l} \int_0^{2l} f(x)^2 \, \mathrm{d}x. \tag{81}$$

Example 9. (§6.14, 5 b)) Expand the following function to a complex Fourier series

$$f(x) = \cosh x, \qquad -\pi < x < \pi. \tag{82}$$

Solution. We have $l = \pi$. Recall that $\cosh x = \frac{e^x + e^{-x}}{2}$. We compute

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^x + e^{-x}}{2} \, \mathrm{d}x = \frac{1}{2\pi} \left(e^{\pi} - e^{-\pi}\right); \tag{83}$$

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^x + e^{-x}}{2} e^{-ikx} \, \mathrm{d}x \\ &= \frac{1}{4\pi} \left[\int_{-\pi}^{\pi} e^{(1-ik)x} \, \mathrm{d}x + \int_{-\pi}^{\pi} e^{-(1+ik)x} \, \mathrm{d}x \right] \\ &= \frac{1}{4\pi} \left[\frac{1}{1-ik} e^{(1-ik)x} |_{-\pi}^{\pi} - \frac{1}{1+ik} e^{-(1+ik)x} |_{-\pi}^{\pi} \right] \\ &= \frac{1}{4\pi} \left[\frac{1}{1-ik} \left(e^{\pi} (-1)^k - e^{-\pi} (-1)^k \right) - \frac{1}{1+ik} \left(e^{-\pi} (-1)^k - e^{\pi} (-1)^k \right) \right] \\ &= \frac{1}{4\pi} \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right) \left(e^{\pi} - e^{-\pi} \right) (-1)^k \\ &= \frac{1}{2\pi} \frac{(-1)^k}{1+k^2} \left(e^{\pi} - e^{-\pi} \right). \end{aligned} \tag{84}$$

Noticing $1 + 0^2 = 1$, the complex Fourier series can be written in the following compact form:

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \frac{(-1)^k}{1+k^2} \left(e^{\pi} - e^{-\pi} \right) e^{ikx}. \tag{85}$$

2.4. Double Fourier series.

We just mention that the theory of Fourier series can be extended to functions with more than one variables. For example, in 2D, instead of $\cos\left(\frac{n\pi}{l} x\right)$ and $\sin\left(\frac{n\pi}{l} x\right)$, we have four combinations

$$\cos\left(\frac{n\pi}{l} x\right) \cos\left(\frac{n\pi}{l} y\right), \ \cos\left(\frac{n\pi}{l} x\right) \sin\left(\frac{n\pi}{l} y\right), \ \sin\left(\frac{n\pi}{l} x\right) \cos\left(\frac{n\pi}{l} y\right), \ \sin\left(\frac{n\pi}{l} x\right) \sin\left(\frac{n\pi}{l} y\right). \tag{86}$$

See §6.12 of the textbook for details.

Remark 10. In higher dimensions, the complex representation becomes more convenient to use, as instead of 2^d combinations of sin's and cos's (d is the dimension), we can simply write the general form of the basis function as

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{87}$$

where $\boldsymbol{k} = \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$.

2.5. Non-periodic functions and Fourier transform.

We have presented a satisfactory theory for periodic functions. Now how about non-periodic functions? Here we give a hint of what happens in this general case. We take the interval to be $(-l, l)$.

Recall the complex Fourier series formulas for f with period $2l$:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi}{l} x}, \tag{88}$$

with

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in\pi x/l} \, \mathrm{d}x. \tag{89}$$

Furthermore we have the Parseval's relation

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2l} \int_{-l}^{l} f(x)^2 \, \mathrm{d}x. \tag{90}$$

Now setting

$$\lambda = \frac{n\,\pi}{l}, \qquad c_\lambda = \frac{l}{\pi}\,c_n, \tag{91}$$

we have

$$\mathrm{d}\lambda = \frac{\pi}{l}\,\mathrm{d}n = \frac{\pi}{l}, \tag{92}$$

and the above formulas can be formally re-written as (taking $\lambda \nearrow \infty$)

$$f(x) = \sum_{-\infty}^{\infty} c_n\,e^{i\frac{n\pi}{l}x} = \int_{-\infty}^{\infty} c_n\,e^{i\lambda x}\,\mathrm{d}n = \int_{-\infty}^{\infty} \frac{\pi}{l}\,c_\lambda\,e^{i\lambda x}\mathrm{d}n = \int_{-\infty}^{\infty} c_\lambda\,e^{i\lambda x}\,\mathrm{d}\lambda, \tag{93}$$

$$c_\lambda = \frac{l}{\pi}\,c_n = \frac{1}{2\,\pi}\int_{-l}^{l} f(x)\,e^{-i\lambda x}\,\mathrm{d}x \to \frac{1}{2\,\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,\mathrm{d}x, \tag{94}$$

$$\int_{-\infty}^{\infty} |c_\lambda|^2\,\mathrm{d}\lambda = \int_{-\infty}^{\infty}\left(\frac{l}{\pi}\right)^2 |c_n|^2\,\frac{\pi}{l}\,\mathrm{d}n = \frac{l}{\pi}\sum |c_n|^2 = \frac{1}{2\,\pi}\int_{-l}^{l} f(x)^2\,\mathrm{d}x \to \frac{1}{2\,\pi}\int_{-\infty}^{\infty} f(x)^2\,\mathrm{d}x. \tag{95}$$

The formulas

$$c_\lambda = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,\mathrm{d}x, \qquad f(x) = \int_{-\infty}^{\infty} c_\lambda\,e^{i\lambda x}\,\mathrm{d}\lambda \tag{96}$$

are called the Fourier transform. It turns out that these formulas, including the Parseval's relation

$$\int_{-\infty}^{\infty} |c_\lambda|^2\,\mathrm{d}\lambda = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} f(x)^2\,\mathrm{d}x \tag{97}$$

can be rigorously derived for a reasonably large class of functions.

For a trignometric version of the Fourier transform theory, see §6.13 of the textbook.

3. Applications of the Fourier series theory.

The theory of Fourier series can be applied to many PDE problems.[2]

3.1. The wave equation.

We return to the problem

$$\begin{aligned} u_{tt} - c^2\,u_{xx} &= 0, \qquad 0 < x < l,\ t > 0 && (98)\\ u(x,0) &= f(x), \quad 0 \leqslant x \leqslant l, && (99)\\ u_t(x,0) &= g(x), \quad 0 \leqslant x \leqslant l, && (100)\\ u(0,t) &= 0, \qquad t \geqslant 0, && (101)\\ u(l,t) &= 0, \qquad t \geqslant 0. && (102)\end{aligned}$$

We have seen that the solution can be represented by

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\left(\cos\frac{n\,\pi\,c}{l}\,t\right) + b_n\sin\left(\frac{n\,\pi\,c}{l}\,t\right)\right)\sin\left(\frac{n\,\pi}{l}\,x\right) \tag{103}$$

with the coefficients determined from

$$\begin{aligned} f(x) &= u(x,0) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\,\pi}{l}\,x\right) && (104)\\ g(x) &= u_t(x,0) = \sum_{n=1}^{\infty} b_n\left(\frac{n\,\pi\,c}{l}\right)\sin\left(\frac{n\,\pi}{l}\,x\right). && (105)\end{aligned}$$

Now we try to compute a_n, b_n.

First we should note a problem here. The formula determining the coefficients for the Fourier series reads

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}x, \tag{106}$$

2. In fact, the Fourier transform theory is the foundation of a complete general theory of linear partial differential equations, the developer L. Hörmander was awarded the Fields Medal due to this contribution.

but our $f(x)$ is only defined on $(0,l)$. Therefore we need to extend $f(x)$ to $(-l,l)$ in an appropriate way. This extension must not produce any cos terms. Therefore we should extend f oddly,

$$\tilde{f}(x)=\begin{cases} f(x) & 0<x<l \\ -f(-x) & -l<x<0 \end{cases} \tag{107}$$

which leads to

$$a_n=\frac{1}{l}\int_{-l}^{l}\tilde{f}(x)\sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x. \tag{108}$$

Similarly we have

$$b_n=\frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x. \tag{109}$$

Example 11. (§7.9 1 a)) Solve

$$\begin{aligned} u_{tt} &= c^2u_{xx}, & 0<x<1,\ t>0 \quad (110)\\ u(x,0) &= x(1-x), & 0\leqslant x\leqslant 1, \quad (111)\\ u_t(x,0) &= 0, & 0\leqslant x\leqslant 1, \quad (112)\\ u(0,t) &= 0, & t\geqslant 0, \quad (113)\\ u(1,t) &= 0, & t\geqslant 0. \quad (114)\end{aligned}$$

Solution. We have $l=1$. Compute

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x \\ &= -\frac{2}{n\pi}\int_0^1 x(1-x)\,\mathrm{d}\cos(n\pi x) \\ &= -\frac{2}{n\pi}\left[x(1-x)\cos(n\pi x)\,|_0^1-\int_0^1\cos(n\pi x)(1-2x)\,\mathrm{d}x\right] \\ &= \frac{2}{(n\pi)^2}\int_0^1(1-2x)\,\mathrm{d}\sin(n\pi x) \\ &= \frac{2}{(n\pi)^2}\left[(1-2x)\sin(n\pi x)\,|_0^1-\int_0^1\sin(n\pi x)(-2)\,\mathrm{d}x\right] \\ &= \frac{4}{(n\pi)^2}\int_0^1\sin(n\pi x)\,\mathrm{d}x \\ &= -\frac{4}{(n\pi)^3}\cos(n\pi x)\,|_0^1 \\ &= \frac{4\left(1-(-1)^n\right)}{(n\pi)^3}. \end{aligned} \tag{115}$$

It is clear that $b_n=0$ as $u_t(x,0)=0$.

Therefore the solution is

$$u(x,t)=\sum_1^{\infty}\frac{4\left(1-(-1)^n\right)}{(n\pi)^3}\cos(n\pi ct)\sin(n\pi x). \tag{116}$$

3.2. The heat equation.

We consider the initial-boundary problem of the hear equation

$$\begin{aligned} u_t-k\,u_{xx} &= 0, & 0<x<l,\ t>0 \quad (117)\\ u(0,t) &= 0, & t\geqslant 0 \quad (118)\\ u(l,t) &= 0, & t\geqslant 0 \quad (119)\\ u(x,0) &= f(x), & 0\leqslant x\leqslant l. \quad (120)\end{aligned}$$

We consider a solution of the form

$$u(x,t)=X(x)\,T(t). \tag{121}$$

Substituting into the equation we obtain

$$X\,T'=k\,X''\,T \implies \frac{T'}{T}=k\,\frac{X''}{X}. \tag{122}$$

Similar to what we have done for the wave equation, we conclude

$$T'=k\,\lambda\,T, \qquad X''=\lambda\,X \tag{123}$$

for some constant λ.

Now considering the boundary condition, we have

$$X''=\lambda\,X, \qquad X(0)=X(l)=0 \tag{124}$$

and consequently (up to a constant factor)

$$X=X_n\equiv\sin\Bigl(\frac{n\,\pi}{l}\,x\Bigr) \tag{125}$$

and (up to a constant factor)

$$T'=-\,k\,\frac{n^2\,\pi^2}{l^2}\,T \implies T(t)=e^{-\left(\frac{n\pi}{l}\right)^2kt}. \tag{126}$$

Thus we expect the general solution to take the form

$$u(x,t)=\sum_{n=1}^{\infty}a_n\,e^{-\left(\frac{n\pi}{l}\right)^2kt}\sin\Bigl(\frac{n\,\pi}{l}\,x\Bigr). \tag{127}$$

where the coefficients

$$a_n=\frac{2}{l}\int_0^l f(x)\sin\Bigl(\frac{n\,\pi}{l}\,x\Bigr)\,\mathrm{d}x.$$

3.3. The Laplace equation.

Consider the problem

$$\begin{aligned}
u_{xx}+u_{yy} &= 0, \qquad 0<x<a,\ 0<x<b, && (128)\\
u(x,0) &= f(x), \quad 0\leqslant x\leqslant a && (129)\\
u(x,b) &= 0, \qquad 0\leqslant x\leqslant a, && (130)\\
u_x(0,y) &= 0, \qquad 0\leqslant y\leqslant b, && (131)\\
u_x(a,y) &= 0, \qquad 0\leqslant y\leqslant b. && (132)
\end{aligned}$$

We solve it using separation of variables.

Consider

$$u(x,y)=X(x)\,Y(y). \tag{133}$$

Substituting into the equation we obtain

$$X''\,Y+X\,Y''=0 \implies \frac{X''}{X}=-\,\frac{Y''}{Y} \tag{134}$$

which leads to

$$X''-\lambda\,X=0, \qquad Y''+\lambda\,Y=0 \tag{135}$$

for some constant λ.

Taking into account the boundary conditions for X, we have

$$X''-\lambda\,X=0,\ X'(0)=X'(a)=0 \implies \lambda=-\Bigl(\frac{n\,\pi}{a}\Bigr)^2,\ X=A_n\cos\Bigl(\frac{n\,\pi}{a}\,x\Bigr) \tag{136}$$

and therefore, when $n\neq 0$,

$$Y''-\Bigl(\frac{n\,\pi}{a}\Bigr)^2Y=0,\ Y(0)\neq 0,\ Y(b)=0 \implies Y=B_n\Bigl[e^{\frac{n\pi}{a}y}-e^{\frac{2n\pi}{a}b}\,e^{-\frac{n\pi}{a}y}\Bigr]=B_n'\sinh\Bigl(\frac{n\,\pi}{a}\,(y-b)\Bigr). \tag{137}$$

When $n=0$,

$$Y''=0,\ Y(0)=1,\ Y(b)=0 \implies Y=B_0\,(1-y/b). \tag{138}$$

Thus the solution should be represented by

$$u(x,y)=\frac{(b-y)}{b}\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\Big(\frac{n\,\pi}{a}\,x\Big)\sinh\Big(\frac{n\,\pi}{a}\,(y-b)\Big). \tag{139}$$

To determine the coefficients, we use

$$f(x)=u(x,0)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\sinh\Big(-\frac{n\,\pi}{a}\,b\Big)\cos\Big(\frac{n\,\pi}{a}\,x\Big) \tag{140}$$

which gives (note that we are expanding into a cosine series here)

$$a_0=\frac{2}{a}\int_0^a f(x)\,\mathrm{d}x, \qquad a_n=\frac{-2}{a\sinh\big(\frac{n\,\pi\,b}{a}\big)}\int_0^a f(x)\cos\Big(\frac{n\,\pi\,x}{a}\Big). \tag{141}$$

3.4. Nonhomogeneous problems.

Now we show the power of the method of separation of variables (more precisely, the power of Fourier series) by studying a problem that is beyond the power of the method of characteristics we have learned.

$$\begin{aligned} u_{tt}-c^2\,u_{xx} &= h(x,t), && 0<x<l,\ t>0 && (142)\\ u(x,0) &= f(x), && 0\leqslant x\leqslant l && (143)\\ u_t(x,0) &= g(x), && 0\leqslant x\leqslant l && (144)\\ u(0,t) &= 0, && t\geqslant 0 && (145)\\ u(l,t) &= 0, && t\geqslant 0. && (146)\end{aligned}$$

As $h(x,t)$ is not separated, it is not possible to find any solution of form $u(x,t)=X(x)\,T(t)$ for the equation

$$u_{tt}-c^2\,u_{xx}=h(x,t). \tag{147}$$

However, it is still possible to solve this problem using our knowledge of Fourier series. We assume the solution takes the form[3]

$$u(x,t)=\sum_{n=1}^{\infty} u_n(t)\sin\Big(\frac{n\,\pi}{l}\,x\Big). \tag{148}$$

Readers familiar with ODE theory may recognize this as a PDE version of the "variation of constants" method. Also note that

We need to represent $h(x,t)$ by $\sin\big(\frac{n\,\pi}{l}\,x\big)$ too.

$$h(x,t)=\sum_{n=1}^{\infty} h_n(t)\sin\Big(\frac{n\,\pi}{l}\,x\Big), \qquad h_n(t)=\frac{2}{l}\int_0^l h(x,t)\sin\Big(\frac{n\,\pi}{l}\,x\Big)\,\mathrm{d}x. \tag{149}$$

Now the equation becomes

$$\sum_{n=1}^{\infty}\Big[u_n''(t)+c^2\Big(\frac{n\,\pi}{l}\Big)^2 u_n(t)\Big]\sin\Big(\frac{n\,\pi}{l}\,x\Big)=\sum_{n=1}^{\infty} h_n(t)\sin\Big(\frac{n\,\pi}{l}\,x\Big). \tag{150}$$

Now it is clear that

$$u_n''(t)+c^2\Big(\frac{n\,\pi}{l}\Big)^2 u_n(t)=h_n(t) \tag{151}$$

with initial conditions

$$u_n(0)=a_n, \qquad u_n'(0)=b_n\Big(\frac{n\,\pi\,c}{l}\Big) \tag{152}$$

3. Note that, as we have $u(0,t)=u(l,t)=0$, Fourier sine series is relevant. Therefore we only keep the $\sin\big(\frac{n\,\pi}{l}\,x\big)$ terms.

where a_n and b_n are from the sine series of f and g:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\,\pi}{l}\,x\right) \Longleftrightarrow a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\,\pi}{l}\,x\right) \mathrm{d}x, \tag{153}$$

$$g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\,\pi\,c}{l}\right) \sin\left(\frac{n\,\pi}{l}\,x\right) \Longleftrightarrow b_n = \frac{2}{n\,\pi\,c}\int_0^l g(x) \sin\left(\frac{n\,\pi}{l}\,x\right) \mathrm{d}x. \tag{154}$$

To solve this equation, we notice that

$$u_n(t) = v_n(t) + w_n(t) \tag{155}$$

where v_n and w_n solve

$$v_n''(t) + \left(\frac{n\,\pi\,c}{l}\right)^2 v_n(t) = 0, \qquad v_n(0) = a_n, \quad v_n'(0) = b_n \left(\frac{n\,\pi\,c}{l}\right), \tag{156}$$

$$w_n''(t) + \left(\frac{n\,\pi\,c}{l}\right)^2 v_n(t) = h_n(t), \qquad w_n(0) = 0, \quad w_n'(0) = 0. \tag{157}$$

The former equation yields

$$v_n(t) = a_n \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + b_n \sin\left(\frac{n\,\pi\,c}{l}\,t\right); \tag{158}$$

To solve the second equation, we need the following Duhamel's principle:

> The solution to the equation
>
> $$v''(t) + \lambda^2\, v(t) = h(t), \qquad v(0) = 0, \quad v'(0) = 0 \tag{159}$$
>
> is given by
>
> $$v(t) = \int_0^t w(t; s)\, \mathrm{d}s \tag{160}$$
>
> where $w(t; s)$ solves
>
> $$w''(t) + \lambda^2\, w(t) = 0, \qquad w(s) = 0, \quad w'(s) = h(s). \tag{161}$$

Using the above principle we have

$$w_n(t) = \left(\frac{n\,\pi\,c}{l}\right)^{-1} \int_0^t h_n(s) \sin\left(\frac{n\,\pi\,c}{l}\,(t-s)\right) \mathrm{d}s. \tag{162}$$

Therefore the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{n\,\pi\,c}{l}\,t\right) + b_n \sin\left(\frac{n\,\pi\,c}{l}\,t\right) + \left(\frac{n\,\pi\,c}{l}\right)^{-1} \int_0^t h_n(s) \sin\left(\frac{n\,\pi\,c}{l}\,(t-s)\right) \mathrm{d}s\right\} \sin\left(\frac{n\,\pi}{l}\,x\right). \tag{163}$$

Example 12. (§7.9 12) Solve the problem

$$\begin{aligned} u_{tt} - c^2\, u_{xx} &= A\,x, & 0 < x < 1,\ t > 0 \quad (164)\\ u(x,0) &= 0, & 0 \leqslant x \leqslant 1 \quad (165)\\ u_t(x,0) &= 0, & 0 \leqslant x \leqslant 1 \quad (166)\\ u(0,t) &= 0, & t > 0 \quad (167)\\ u(1,t) &= 0, & t > 0 \quad (168) \end{aligned}$$

Solution. Recall that we should use the formula

$$u(x, t) = \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{n\,\pi\,c}{l}\,x\right) + b_n \sin\left(\frac{n\,\pi\,c}{l}\,x\right) + \left(\frac{n\,\pi\,c}{l}\right)^{-1} \int_0^t h_n(s) \sin\left(\frac{n\,\pi\,c}{l}\,(t-s)\right) \mathrm{d}s\right\} \sin\left(\frac{n\,\pi}{l}\,x\right). \tag{169}$$

with

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}x, b_n = \frac{2}{n\,\pi\,c}\int_0^l g(x)\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}x, h_n(t) = \frac{2}{l}\int_0^l h(x,t)\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}x. \tag{170}$$

Now that $f = g = 0$, $h(x,t) = A\,x$, $l = 1$, we have $a_n = b_n = 0$ and

$$\begin{aligned}
h_n(t) &= 2\int_0^1 A\,x\sin(n\,\pi\,x)\,\mathrm{d}x \\
&= -2\,A\,\frac{1}{n\,\pi}\int_0^1 x\,\mathrm{d}\cos(n\,\pi\,x) \\
&= -\frac{2\,A}{n\,\pi}\left[x\cos(n\,\pi\,x)\,|_0^1 - \int_0^1 \cos(n\,\pi\,x)\,\mathrm{d}x\right] \\
&= \frac{2\,A\,(-1)^{n+1}}{n\,\pi}.
\end{aligned} \tag{171}$$

Therefore

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left\{\frac{1}{n\,\pi\,c}\int_0^t \frac{2\,A\,(-1)^{n+1}}{n\,\pi}\sin(n\,\pi\,c\,(t-s))\,\mathrm{d}s\right\}\sin(n\,\pi\,x) \\
&= \sum_{n=1}^{\infty}\left\{\left(\frac{1}{n\,\pi\,c}\right)^2\frac{2\,A\,(-1)^{n+1}}{n\,\pi}\cos(n\,\pi\,c\,(t-s))\,|_0^t\right\}\sin(n\,\pi\,x) \\
&= \sum_{n=1}^{\infty}\frac{2\,A\,(-1)^{n+1}}{n^3\,\pi^3\,c^2}\,[1-\cos(n\,\pi\,c\,t)]\sin(n\,\pi\,x).
\end{aligned} \tag{172}$$

We can check

$$\begin{aligned}
u_{tt} - c^2\,u_{xx} &= \sum_{n=1}^{\infty}\left\{\frac{2\,A\,(-1)^{n+1}}{n\,\pi}\cos(n\,\pi\,c\,t)\sin(n\,\pi\,x)\right\} \\
&\quad - c^2\sum_{n=1}^{\infty}\left[-\frac{2\,A\,(-1)^{n+1}}{n\,\pi\,c^2}\,[1-\cos(n\,\pi\,c\,t)]\sin(n\,\pi\,x)\right] \\
&= \sum_{n=1}^{\infty}\frac{2\,A\,(-1)^{n+1}}{n\,\pi}\sin(n\,\pi\,x) = A\,x.
\end{aligned} \tag{173}$$