Week 03: Classification of Second-Order Linear Equations

In last week's lectures we have illustrated how to obtain the general solutions of first order PDEs using the method of characteristics. We will try to do the same thing for second order PDEs. It turns out that only a small portion of linear 2nd order PDEs can be solved in the sense of obtaining general solutions.

For simplicity, we will only consider 2nd order equations in two independent variables, whose general form is

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F u = G. \tag{1}$$

where each coefficient $A, B, C, \ldots$ is a function of x, y.

It turns out that one can simplify the 2nd order terms to one of the following three so-called "canonical" forms

1. hyperbolic: u_{xy} or $u_{xx} - u_{yy}$;
2. parabolic: u_{xx} or u_{yy};
3. elliptic: $u_{xx} + u_{yy}$.

Not only the method of finding solutions, but also the properties of the equations/solutions are very different for each category.

1. Reduction to canonical forms.

1.1. General strategy.

The idea is to apply a change of variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y). \tag{2}$$

This gives

$$\begin{aligned}
u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x, & (3)\\
u_y &= u_\xi\,\xi_y + u_\eta\,\eta_y, & (4)\\
u_{xx} &= u_{\xi\xi}\,\xi_x^2 + 2\,u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx}, & (5)\\
u_{xy} &= u_{\xi\xi}\,\xi_x\,\xi_y + u_{\xi\eta}\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{xy} + u_\eta\,\eta_{xy}, & (6)\\
u_{yy} &= u_{\xi\xi}\,\xi_y^2 + 2\,u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}\,\eta_y^2 + u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy}. & (7)
\end{aligned}$$

Remark 1. How to remember the above formulas:

Substituting these into the equation we obtain

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^*\,u_{\eta\eta} + D^* u_\xi + E^*\,u_\eta + F^*\,u = G^*. \tag{8}$$

with

$$\begin{aligned}
A^* &= A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\,\xi_y^2, & (9)\\
B^* &= 2\,A\,\xi_x\,\eta_y + B\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + 2\,C\,\xi_y\,\eta_y, & (10)\\
C^* &= A\,\eta_x^2 + B\,\eta_x\,\eta_y + C\,\eta_y^2, & (11)\\
D^* &= A\,\xi_{xx} + B\,\xi_{xy} + C\,\xi_{yy} + D\,\xi_x + E\,\xi_y, & (12)\\
E^* &= A\,\eta_{xx} + B\,\eta_{xy} + C\,\eta_{yy} + D\,\eta_x + E\,\eta_y, & (13)\\
F^* &= F, & (14)\\
G^* &= G. & (15)
\end{aligned}$$

Now recall that our purpose is to reduce the equation to canonical form. In other words, we would explore the possibility of choosing appropriate ξ, η such that

1. $A^* = C^* = 0$, $B^* \neq 0$, or
2. $B^* = 0$, exactly one of A^* and $C^* = 0$, or
3. $A^* = C^* \neq 0$, $B^* = 0$.

Remark 2. Can it happen that $A^* = B^* = C^* = 0$? In principle, it is possible. But that just means that the equation is in fact just first order.

1.2. Hyperbolic case.

If the equation can be reduced to the hyperbolic canonical form, then we should be able to find ξ, η such that

$$A^* = A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\,\xi_y^2 = 0 \implies A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0, \tag{16}$$

and

$$C^* = A\,\eta_x^2 + B\,\eta_x\,\eta_y + C\,\eta_y^2 = 0 \implies A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0. \tag{17}$$

We have

1. Since ξ, η must be independent, the equation

$$A\,\zeta^2 + B\,\zeta + C = 0 \tag{18}$$

   must admit two different solutions. As a consequence the equation can be reduced to the hyperbolic canonical form only when

$$B^2 - 4\,A\,C > 0. \tag{19}$$

2. Consider the function $\xi = \xi(x, y)$, what does the ratio $r = \xi_x / \xi_y$ tell us? One easily sees

$$\xi_x - r\,\xi_y = 0. \tag{20}$$

   But this is just a first order PDE for ξ! Thus we can solve it (and obtain ξ) using the method of characteristics:

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-r} = \frac{\mathrm{d}u}{0} \tag{21}$$

   which can be simplified to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -r = -\frac{\xi_x}{\xi_y}. \tag{22}$$

3. Integrating this, we can obtain ξ and similarly η.

From the above discussion we see that when $B^2 - 4\,A\,C > 0$, we can obtain ξ and η and reduce the equation to

$$u_{\xi\eta} = H. \tag{23}$$

If we let

$$\alpha = \xi + \eta, \quad \beta = \xi - \eta, \tag{24}$$

the equation becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = H_1. \tag{25}$$

1.3. Parabolic case.

In this case we obtain $B^* = 0$ and one of $A^*, C^* = 0$. This gives

$$(B^*)^2 - 4\,A^*\,C^* = 0. \tag{26}$$

One can check that as a consequence

$$B^2 - 4\,A\,C = 0. \tag{27}$$

which means we can only obtain one function (either ξ or η) by solving

$$A\,\zeta_x^2 + B\,\zeta_x\,\zeta_y + C\,\zeta_y^2 = 0. \tag{28}$$

Note that onece $A^* = 0$, B^* has to be 0.

The canonical form is

$$u_{\xi\xi} = H_2(\xi, \eta, u, u_\xi, u_\eta). \tag{29}$$

1.4. Elliptic case.

The remaining case is $B^2 - 4AC < 0$. In this case we can obtain two complex roots. In other words

$$\frac{\xi_x}{\xi_y} = \left(\frac{\eta_x}{\eta_y}\right)^*. \tag{30}$$

We see that this holds when ξ and η are complex conjugates.

In this case, we can introduce new variables

$$\alpha = \frac{1}{2}(\xi + \eta), \qquad \beta = \frac{1}{2i}(\xi - \eta) \tag{31}$$

or equivalently

$$\xi = \alpha + \beta i, \quad \eta = \alpha - \beta i. \tag{32}$$

Using the change of variables formula we have

$$u_{\alpha\alpha} + u_{\beta\beta} = 4 u_{\xi\eta} \tag{33}$$

as a consequence, the canonical form in real variables α, β is

$$u_{\alpha\alpha} + u_{\beta\beta} = H_3(\alpha, \beta, u, u_\alpha, u_\beta). \tag{34}$$

Remark 3. In practice, it is easier to obtain $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ and then transform the equation, without first reducing the equation into $u_{\xi\eta} = H_4$.

1.5. Examples.

Summary: To solve a 2nd order linear PDE, we follow the following steps.

1. Solve $A\, u_{xx} + B\, u_{xy} + C\, u_{yy} \implies A\,(\mathrm{d}y)^2 - B\,(\mathrm{d}x)\,(\mathrm{d}y) + C\,(\mathrm{d}x)^2 = 0$. Note the sign change.[1] Obtain
   a) new variables ξ and η when $B^2 - 4AC > 0$ (hyperbolic);
   b) new variable ξ, choose any η with $\det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \neq 0$, when $B^2 - 4AC = 0$ (parabolic);
   c) two complex functions ξ and η, set $\alpha = (\xi + \eta)/2$, $\beta = (\xi - \eta)/2\,i$ as new variables, when $B^2 - 4AC < 0$ (elliptic).
2. Perform change of variables and reduce the equation to canonical forms using the following formulas:

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x, & (39)\\
u_y &= u_\xi \xi_y + u_\eta \eta_y, & (40)\\
u_{xx} &= u_{\xi\xi} \xi_x^2 + 2 u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, & (41)\\
u_{xy} &= u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}, & (42)\\
u_{yy} &= u_{\xi\xi} \xi_y^2 + 2 u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}. & (43)
\end{aligned}$$

3. Try to obtain the general solution of the canonical form equation.
4. (For Cauchy problems) Substitute the Cauchy data into the general solution and determine the solution.

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1. We explain a bit. Remember that $\frac{\xi_x}{\xi_y}$ and $\frac{\eta_x}{\eta_y}$ solve

$$A r^2 + B r + C = 0. \tag{35}$$

Let r_1, r_2 be the two roots, then the equation can be written as

$$A(r - r_1)(r - r_2) = 0 \implies B = -A(r_1 + r_2), \quad C = A r_1 r_2. \tag{36}$$

Now as $\frac{\mathrm{d}y}{\mathrm{d}x} = -r_1, -r_2$, they solve the equation

$$A(r + r_1)(r + r_2) = 0 \iff A r^2 - B r + C = 0. \tag{37}$$

In other words, $\mathrm{d}y, \mathrm{d}x$ satisfy

$$A(\mathrm{d}y)^2 - B(\mathrm{d}x)(\mathrm{d}y) + C(\mathrm{d}x)^2 = 0. \tag{38}$$

Example 4. (§4.6, 1) Determine the region in which the given equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form.

- **(a)**

$$x\,u_{xx}+u_{yy}=x^2. \tag{44}$$

Solution. We have $A=x, B=0, C=1$. Thus

$$B^2-4\,A\,C=-\,4\,x. \tag{45}$$

  - $x<0$: Hyperbolic.
    We solve

$$x\,(\mathrm{d}y)^2+(\mathrm{d}x)^2=0. \tag{46}$$

This gives

$$\mathrm{d}x\pm\sqrt{-\,x}\,\mathrm{d}y=0 \tag{47}$$

which leads to

$$\mathrm{d}\left[\,y\pm 2\,\sqrt{-\,x}\,\right]=0 \tag{48}$$

therefore

$$\xi=y+2\,\sqrt{-\,x}, \qquad \eta=y-2\,\sqrt{-\,x} \tag{49}$$

which gives

$$\xi_x=-\frac{1}{\sqrt{-\,x}},\ \ \xi_y=1,\ \ \xi_{xx}=-\frac{1}{2\left(\sqrt{-\,x}\right)^3},\ \ \xi_{xy}=0,\ \ \xi_{yy}=0; \tag{50}$$

$$\eta_x=\frac{1}{\sqrt{-\,x}},\ \ \eta_y=1,\ \ \eta_{xx}=\frac{1}{2\left(\sqrt{-\,x}\right)^3},\ \ \eta_{xy}=0,\ \ \eta_{yy}=0. \tag{51}$$

We compute

$$u_{xx} = u_{\xi\xi}\,\xi_x^2+2\,u_{\xi\eta}\,\xi_x\,\eta_x+u_{\eta\eta}\,\eta_x^2+u_\xi\,\xi_{xx}+u_\eta\,\eta_{xx}=-\frac{u_{\xi\xi}}{x}+\frac{2\,u_{\xi\eta}}{x}-\frac{u_{\eta\eta}}{x}+\frac{u_\eta-u_\xi}{2\left(\sqrt{-\,x}\right)^3}, \tag{52}$$

$$u_{xy} = u_{\xi\xi}\,\xi_x\,\xi_y+u_{\xi\eta}\,(\xi_x\,\eta_y+\xi_y\,\eta_x)+u_{\eta\eta}\,\eta_x\,\eta_y+u_\xi\,\xi_{xy}+u_\eta\,\eta_{xy}=\frac{u_{\eta\eta}-u_{\xi\xi}}{\sqrt{-\,x}}, \tag{53}$$

$$u_{yy} = u_{\xi\xi}\,\xi_y^2+2\,u_{\xi\eta}\,\xi_y\,\eta_y+u_{\eta\eta}\,\eta_y^2+u_\xi\,\xi_{yy}+u_\eta\,\eta_{yy}=u_{\xi\xi}+2\,u_{\xi\eta}+u_{\eta\eta}. \tag{54}$$

Thus the equation becomes

$$4\,u_{\xi\eta}+\frac{u_\xi-u_\eta}{2\,\sqrt{-\,x}}=x^2. \tag{55}$$

From the change of variables we obtain

$$\sqrt{-\,x}=\frac{\xi-\eta}{4} \tag{56}$$

and as a consequence

$$u_{\xi\eta}=\frac{1}{4}\left(\frac{\xi-\eta}{4}\right)^4-\frac{1}{2}\left(\frac{1}{\xi-\eta}\right)(u_\xi-u_\eta). \tag{57}$$

  - $x=0$: parabolic. In this case the equation becomes

$$u_{yy}=x^2 \tag{58}$$

which is already in canonical form.

  - $x>0$: elliptic. In this case we still have

$$x\,(\mathrm{d}y)^2+(\mathrm{d}x)^2=0. \tag{59}$$

which gives

$$\pm i\sqrt{x}\,\mathrm{d}y + \mathrm{d}x = 0 \Longrightarrow \mathrm{d}\big[2\sqrt{x} \pm i\,y\big] = 0. \tag{60}$$

Thus

$$\xi = 2\sqrt{x} + i\,y, \qquad \eta = 2\sqrt{x} - i\,y. \tag{61}$$

We then have

$$\alpha = \frac{\xi + \eta}{2} = 2\sqrt{x}, \qquad \beta = \frac{\xi - \eta}{2\,i} = y. \tag{62}$$

This leads to

$$\alpha_x = \frac{1}{\sqrt{x}},\ \ \alpha_y = 0,\ \ \alpha_{xx} = -\frac{1}{2\left(\sqrt{x}\right)^3},\ \ \alpha_{xy} = 0,\ \ \alpha_{yy} = 0, \tag{63}$$

$$\beta_x = 0,\ \ \beta_y = 1,\ \ \beta_{xx} = \beta_{xy} = \beta_{yy} = 0. \tag{64}$$

Consequently

$$u_{xx} = \frac{u_{\alpha\alpha}}{x} - \frac{u_\alpha}{2\sqrt{x}^3}, \qquad u_{yy} = u_{\beta\beta} \tag{65}$$

and the equation becomes

$$u_{\alpha\alpha} + u_{\beta\beta} = x^2 + \frac{u_\alpha}{2\sqrt{x}} = \frac{u_\alpha}{\alpha} + \left(\frac{\alpha}{2}\right)^4. \tag{66}$$

- **(d)**

$$x^2\,u_{xx} - 2\,x\,y\,u_{xy} + y^2\,u_{yy} = e^x. \tag{67}$$

Solution. We have

$$B^2 - 4\,A\,C \equiv 0 \tag{68}$$

So the equation is of parabolic type. We solve the characteristics equation

$$x^2\,(\mathrm{d}y)^2 - 2\,x\,y\,\mathrm{d}x\,\mathrm{d}y + y^2\,(\mathrm{d}x)^2 = 0 \tag{69}$$

which reduces to

$$(x\,\mathrm{d}y + y\,\mathrm{d}x)^2 = 0 \Longrightarrow \xi = x\,y. \tag{70}$$

Thus the Jacobian is

$$\det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \det\begin{pmatrix} y & x \\ \eta_x & \eta_y \end{pmatrix}. \tag{71}$$

We can take for example $\eta = x$ to make the Jacobian nonzero. We have

$$\xi_x = y,\ \ \xi_y = x,\ \ \xi_{xx} = 0,\ \ \xi_{xy} = 1,\ \ \xi_{yy} = 0, \tag{72}$$

$$\eta_x = 1,\ \ \eta_y = \eta_{xx} = \eta_{xy} = \eta_{yy} = 0. \tag{73}$$

Thus we have

$$u_{xx} = y^2\,u_{\xi\xi} + 2\,y\,u_{\xi\eta} + x^2\,u_{\eta\eta},\ \ u_{xy} = x\,y\,u_{\xi\xi} + x\,u_{\xi\eta} + u_\xi,\ \ u_{yy} = x^2\,u_{\xi\xi} \tag{74}$$

which leads to

$$x^4\,u_{\eta\eta} - 2\,x\,y\,u_\xi = e^x. \tag{75}$$

So the canonical form is

$$u_{\eta\eta} = \frac{2\,\xi}{\eta^4}\,u_\xi + \frac{1}{\eta^4}\,e^\eta. \tag{76}$$

2. Equations with constant coefficients.

When the coefficients A—F are constants, sometimes it is possible to find the general solution after reduction to canonical forms. In particular, we can obtain general solutions when D—$G = 0$.

Example 5. Find the general solution of the 2nd order PDE

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} = 0 \tag{77}$$

where A, B, C are constants.[2]

Solution. We deal with the three cases individually.

1. $B^2 - 4AC > 0$. We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_{1,2} \equiv \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{78}$$
which gives
$$\xi = y - \lambda_1 x, \quad \eta = y - \lambda_2 x \tag{79}$$
The equation becomes
$$u_{\xi\eta} = 0 \tag{80}$$
whose general solutions are
$$u = \phi(\xi) + \psi(\eta) = \phi(y - \lambda_1 x) + \psi(y - \lambda_2 x). \tag{81}$$
(There is a typo in the book here).

In the case $A = 0$, we use ξ_y/ξ_x instead of ξ_x/ξ_y to obtain
$$B\left(\frac{\xi_y}{\xi_x}\right) + C\left(\frac{\xi_y}{\xi_x}\right)^2 = 0 \Longrightarrow \xi = x, \quad \eta = x - \frac{B}{C} y. \tag{82}$$

2. $B^2 - 4AC = 0$. In this case from the equation we can only obtain
$$\xi = y - \frac{B}{2A} x. \tag{83}$$
It turns out that η can be chosen arbitrarily as long as the Jacobian
$$J = \det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \neq 0. \tag{84}$$
The canonical form is
$$u_{\eta\eta} = 0 \tag{85}$$
whose general solutions are
$$u = \phi(\xi) + \eta\,\psi(\xi). \tag{86}$$
When $B \neq 0$, one can simply choose $\eta = y$ and therefore
$$u = \phi\left(y - \frac{B}{2A} x\right) + y\,\psi\left(y - \frac{B}{2A} x\right). \tag{87}$$

3. $B^2 - 4AC < 0$. We obtain
$$\xi = y - (a + i\,b)\,x, \qquad \eta = y - (a - i\,b)\,x, \tag{88}$$
with
$$a = \frac{B}{2A}, \qquad b = \frac{1}{2A}\sqrt{4AC - B^2}. \tag{89}$$
As a consequence
$$\alpha = y - a\,x, \qquad \beta = -b\,x. \tag{90}$$
Note that basically the only second order equation we can solve is
$$u_{\xi\eta} = 0. \tag{91}$$
In this case, we have
$$u = \phi(\xi) + \psi(\eta) = \phi((y - a\,x) - i\,b\,x) + \psi((y - a\,x) + i\,b\,x). \tag{92}$$

Example 6. (§4.6, 2(iii)) Obtain the general solution of the following equation:
$$4\,u_{xx} + 12\,u_{xy} + 9\,u_{yy} - 9\,u = 9. \tag{93}$$

2. Such an equation is called the *Euler equation*.

(Note that there is a typo in the book)

Solution. First we reduce it to canonical form. As $B^2 - 4\,A\,C = 0$, the equation is parabolic. The characteristics equation is

$$4\,(\mathrm{d}y)^2 + 12\,(\mathrm{d}x)\,(\mathrm{d}y) + 9\,(\mathrm{d}x)^2 = 0 \Longrightarrow 2\,\mathrm{d}y - 3\,\mathrm{d}x = 0. \tag{94}$$

Thus we have

$$\xi = 2\,y - 3\,x. \tag{95}$$

We can simply take $\eta = y$.

Thus

$$\xi_x = -3,\ \ \xi_y = 2,\ \ \xi_{xx} = \xi_{xy} = \xi_{yy} = 0; \qquad \eta_y = 1,\ \ \eta_x = \eta_{xx} = \eta_{xy} = \eta_{yy} = 0. \tag{96}$$

Under this change of variables, we have

$$u_{xx} = u_{\xi\xi}\,\xi_x^2 + 2\,u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx} = 9\,u_{\xi\xi}; \tag{97}$$

$$u_{xy} = u_{\xi\xi}\,\xi_x\,\xi_y + u_{\xi\eta}\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{xy} + u_\eta\,\eta_{xy} = -6\,u_{\xi\xi} - 3\,u_{\xi\eta}; \tag{98}$$

$$u_{yy} = u_{\xi\xi}\,\xi_y^2 + 2\,u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}\,\eta_y^2 + u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy} = 4\,u_{\xi\xi} + 4\,u_{\xi\eta} + u_{\eta\eta}; \tag{99}$$

Thus the equation reduces to

$$9\,u_{\eta\eta} - 9\,u = 9 \iff u_{\eta\eta} - u = 1. \tag{100}$$

We see that the general solution is

$$u(\xi,\eta) = f(\xi)\,e^{\eta} + g(\xi)\,e^{-\eta} - 1. \tag{101}$$

Or in (x, y) variables

$$u(x,y) = f(2\,y - 3\,x)\,e^{y} + g(2\,y - 3\,x)\,e^{-y} - 1. \tag{102}$$

Example. (§4.6, 2(iv)) Obtain the general solution of the following equation:

$$u_{xx} + u_{xy} - 2\,u_{yy} - 3\,u_x - 6\,u_y = 9\,(2\,x - y). \tag{103}$$

Solution. We compute

$$B^2 - 4\,A\,C = 1 - 4\,(-2) = 9 > 0 \tag{104}$$

thus the equation is hyperbolic. The characteristics equation is

$$(\mathrm{d}y)^2 - (\mathrm{d}x)\,(\mathrm{d}y) - 2\,(\mathrm{d}x)^2 = 0 \iff (\mathrm{d}y - 2\,\mathrm{d}x)\,(\mathrm{d}y + \mathrm{d}x) = 0 \tag{105}$$

which gives

$$\xi = y - 2\,x, \qquad \eta = y + x. \tag{106}$$

We have

$$\xi_x = -2,\ \ \xi_y = 1; \qquad \eta_x = \eta_y = 1 \tag{107}$$

and all second order derivatives are 0. As a consequence

$$\begin{aligned}
u_{xx} &= u_{\xi\xi}\,\xi_x^2 + 2\,u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx} = 4\,u_{\xi\xi} - 4\,u_{\xi\eta} + u_{\eta\eta}, && (108)\\
u_{xy} &= u_{\xi\xi}\,\xi_x\,\xi_y + u_{\xi\eta}\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{xy} + u_\eta\,\eta_{xy} = -2\,u_{\xi\xi} - u_{\xi\eta} + u_{\eta\eta}, && (109)\\
u_{yy} &= u_{\xi\xi}\,\xi_y^2 + 2\,u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}\,\eta_y^2 + u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy} = u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta}, && (110)\\
u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x = -2\,u_\xi + u_\eta, && (111)\\
u_y &= u_\xi\,\xi_y + u_\eta\,\eta_y = u_\xi + u_\eta, && (112)\\
2\,x - y &= -\xi. && (113)
\end{aligned}$$

The equation reduces to

$$-9\,u_{\xi\eta} - 9\,u_\eta = -9\,\xi \iff (u_\xi + u)_\eta = u_{\xi\eta} - u_\eta = \xi. \tag{114}$$

The general solution can be obtained via

$$u_\xi + u = \xi\,\eta + h(\xi) \implies \left(e^\xi\,u\right)_\xi = e^\xi\,\xi\,\eta + e^\xi\,h(\xi) \implies e^\xi\,u = \eta\,e^\xi\,(\xi - 1) + f(\xi) + g(\eta). \tag{115}$$

Therefore

$$u(\xi,\eta) = \eta\,(\xi - 1) + f(\xi) + g(\eta)\,e^{-\xi} \tag{116}$$

and

$$u(x,y) = (y + x)\,(y - 2\,x - 1) + f(y - 2\,x) + g(y + x)\,e^{2x-y} \tag{117}$$

where f, g are arbitrary functions.

3. Finding general solutions for non-constant coefficient equations.

3.1. Examples.

Example 7. (§4.6, 2(i)) Obtain the general solution.

$$x^2\,u_{x\,x} + 2\,x\,y\,u_{x\,y} + y^2\,u_{y\,y} + x\,y\,u_x + y^2\,u_y = 0. \tag{118}$$

Solution. We check $B^2 - 4\,A\,C = (2\,x\,y)^2 - 4\,x^2\,y^2 = 0$ so the equation is parabolic. The characteristics equation is

$$x^2\,(\mathrm{d}y)^2 - 2\,x\,y\,(\mathrm{d}x)\,(\mathrm{d}y) + y^2\,(\mathrm{d}x)^2 = 0 \implies x\,\mathrm{d}y - y\,\mathrm{d}x = 0. \tag{119}$$

Thus

$$\xi = \frac{y}{x}. \tag{120}$$

We compute

$$J = \det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \det\begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \eta_x & \eta_y \end{pmatrix} \tag{121}$$

and we can take $\eta = x$ to guarantee $J \neq 0$. Now we have

$$\xi_x = -\frac{y}{x^2},\ \ \xi_y = \frac{1}{x},\ \ \xi_{x\,x} = \frac{2\,y}{x^3},\ \ \xi_{x\,y} = -\frac{1}{x^2},\ \ \xi_{y\,y} = 0. \tag{122}$$

$$\eta_x = 1,\ \ \eta_y = \eta_{x\,x} = \eta_{x\,y} = \eta_{y\,y} = 0. \tag{123}$$

This gives

$$\begin{aligned}
u_{x\,x} &= u_{\xi\xi}\,\xi_x^2 + 2\,u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{x\,x} + u_\eta\,\eta_{x\,x} = \frac{y^2}{x^4}\,u_{\xi\xi} - \frac{2\,y}{x^2}\,u_{\xi\eta} + u_{\eta\eta} + \frac{2\,y}{x^3}\,u_\xi, && (124)\\
u_{x\,y} &= u_{\xi\xi}\,\xi_x\,\xi_y + u_{\xi\eta}\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{x\,y} + u_\eta\,\eta_{x\,y} = -\frac{y}{x^3}\,u_{\xi\xi} + \frac{1}{x}\,u_{\xi\eta} - \frac{1}{x^2}\,u_\xi, && (125)\\
u_{y\,y} &= u_{\xi\xi}\,\xi_y^2 + 2\,u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}\,\eta_y^2 + u_\xi\,\xi_{y\,y} + u_\eta\,\eta_{y\,y} = \frac{1}{x^2}\,u_{\xi\xi}, && (126)\\
u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x = -\frac{y}{x^2}\,u_\xi + u_\eta, && (127)\\
u_y &= u_\xi\,\xi_y + u_\eta\,\eta_y = \frac{1}{x}\,u_\xi. && (128)
\end{aligned}$$

The equation becomes

$$x^2\,u_{\eta\eta} + x\,y\,u_\eta = 0 \implies u_{\eta\eta} + \xi\,u_\eta = 0. \tag{129}$$

We solve the equation

$$u_{\eta\eta} + \xi\,u_\eta = 0 \implies u_\eta + \xi\,u = h(\xi) \implies \left(e^{\xi\eta}\,u\right)_\eta = e^{\xi\eta}\,h(\xi) \tag{130}$$

which leads to

$$e^{\xi\eta}\,u = \xi^{-1}\,e^{\xi\eta}\,h(\xi) + g(\xi) \implies u(\xi,\eta) = \xi^{-1}\,h(\xi) + e^{-\xi\eta}\,g(\xi). \tag{131}$$

So finally

$$u(x,y) = f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)e^{-y} \tag{132}$$

with f, g arbitrary functions.

Example 8. (§4.6, 2(ii)) Obtain the general solution.

$$r\,u_{tt} - c^2\,r\,u_{rr} - 2\,c^2\,u_r = 0 \tag{133}$$

where c is a constant.

Solution. We check $B^2 - 4\,A\,C = 0 + 4\,c^2\,r^2 > 0$ so that equation is hyperbolic. The characteristics equation is

$$r\,(\mathrm{d}r)^2 - c^2\,r\,(\mathrm{d}t)^2 = 0 \Longrightarrow \mathrm{d}r \pm c\,\mathrm{d}t = 0 \tag{134}$$

so we take

$$\xi = r + c\,t, \qquad \eta = r - c\,t. \tag{135}$$

From this we obtain

$$\xi_r = 1,\ \ \xi_t = c; \qquad \eta_r = 1,\ \ \eta_t = -\,c \tag{136}$$

and all second order derivatives are zero.

Now we compute

$$\begin{aligned} u_{tt} = u_{\xi\xi}\,\xi_t^2 + 2\,u_{\xi\eta}\,\xi_t\,\eta_t + u_{\eta\eta}\,\eta_t^2 + u_\xi\,\xi_{tt} + u_\eta\,\eta_{tt} &= c^2\,u_{\xi\xi} - 2\,c^2\,u_{\xi\eta} + c^2\,u_{\eta\eta}, & (137)\\ u_{rr} = u_{\xi\xi}\,\xi_r^2 + 2\,u_{\xi\eta}\,\xi_r\,\eta_r + u_{\eta\eta}\,\eta_r^2 + u_\xi\,\xi_{rr} + u_\eta\,\eta_{rr} &= u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta} & (138)\\ u_r = u_\xi\,\xi_r + u_\eta\,\eta_r &= u_\xi + u_\eta. & (139) \end{aligned}$$

The equation then reduces to

$$-\,4\,r\,c^2\,u_{\xi\eta} - 2\,c^2\,u_\xi - 2\,c^2\,u_\eta = 0 \Longrightarrow 2\,r\,u_{\xi\eta} + u_\xi + u_\eta = 0 \Longrightarrow (\xi + \eta)\,u_{\xi\eta} + u_\xi + u_\eta = 0. \tag{140}$$

It turns out that the equation can be rewritten to

$$[(\xi + \eta)\,u]_{\xi\eta} = 0. \tag{141}$$

Therefore the general solutions are

$$u(\xi, \eta) = (\xi + \eta)^{-1}\,[f(\xi) + g(\eta)] \tag{142}$$

or equivalently

$$u(r, t) = r^{-1}\,[f(r + c\,t) + g(r - c\,t)]. \tag{143}$$

Remark 9. In fact, if we let $v = r\,u$ from the very start, we can reduce the equation immediately to the wave equation.

3.2. Further simplifications.

For the hyperbolic case with constant coefficients

$$u_{rs} = a_1\,u_r + a_2\,u_s + a_3\,u + f_1 \tag{144}$$

we can introduce

$$v = u\,e^{-(a_1 r + a_2 s)} \Longrightarrow u = v\,e^{(a_1 r + a_2 s)}. \tag{145}$$

which yields

$$v_{rs} = (a_1\,a_2 + a_3)\,v + g_1 \tag{146}$$

where $g_1 = f_1\,e^{-(a_2 r + a_1 s)}$ when we choose $b = a_1, a = a_2$.

Similarly, by choose appropriate a, b and let $v = u\,e^{-(ar + bs)}$, one can cancel the first order terms and reduce

$$u_{rr} - u_{ss} = a_1^*\,u_r + a_2^*\,u_s + a_3^*\,u + f_1^* \Longrightarrow v_{rr} - v_{ss} = h_1^*\,v + g_1^*, \tag{147}$$

$$u_{ss} = b_1\,u_r + b_2\,u_s + b_3\,u + f_2 \Longrightarrow v_{ss} = h_2\,v + g_2, \tag{148}$$

$$u_{rr} + u_{ss} = c_1\,u_r + c_2\,u_s + c_3\,u + f_3 \Longrightarrow v_{rr} + v_{ss} = h_3\,v + g_3. \tag{149}$$

Remark 10. Note that when the coefficients are not constants, the above trick does not quite work.