(Nov. 9, 12:00pm – 12:50pm, DP6069)

Math 334 2011 Midterm 2 Solutions

Name ________________________

ID# ________________________

Signature ________________________

- Only pen/pencil/eraser are allowed. Scratch papers will be provided.
- Please **write clearly**, with intermediate steps to **show sufficient work** even if you can solve the problem in "one go". Otherwise you may not receive full credit.
- Please box, underline, or highlight the most important parts of your answers.

Problem	Points	Score
1	30	
2	20	
3	15	
4	15	
5	15	
6	5	
Total	100	

Problem 1. (30 pts) Find the first five nonzero terms in the solution of the problem

$$y'' - x\,y' - y = 0, \qquad y(0) = 2, \quad y'(0) = 1. \tag{1}$$

Solution. Write

$$y = \sum_{n=0}^{\infty} a_n\,x^n. \tag{2}$$

Substitute into the equation:

$$\begin{aligned} 0 &= \left(\sum_{n=0}^{\infty} a_n\,x^n\right)'' - x\left(\sum_{n=0}^{\infty} a_n\,x^n\right)' - \left(\sum_{n=0}^{\infty} a_n\,x^n\right) && (3)\\ &= \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} - \sum_{n=0}^{\infty} a_n\,x^n && (4)\\ &= \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n - \sum_{n=1}^{\infty} n\,a_n\,x^n - \sum_{n=0}^{\infty} a_n\,x^n && (5)\\ &= 2\,a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)\,(n+1)\,a_{n+2} - (n+1)\,a_n]\,x^n. && (6) \end{aligned}$$

Thus the recurrence relations are

$$\begin{aligned} 2\,a_2 - a_0 &= 0, && (7)\\ (n+2)\,a_{n+2} - a_n &= 0. && (8) \end{aligned}$$

Now the initial conditions give

$$y(0) = 2 \Longrightarrow a_0 = 2; \qquad y'(0) = 1 \Longrightarrow a_1 = 1. \tag{9}$$

We compute

$$\begin{aligned} (n=0) \qquad a_2 &= \frac{a_0}{2} = 1; && (10)\\ (n=1) \qquad a_3 &= \frac{a_1}{3} = \frac{1}{3}; && (11)\\ (n=2) \qquad a_4 &= \frac{a_2}{4} = \frac{1}{4}. && (12) \end{aligned}$$

We already have five nonzero terms:

$$y(x) = 2 + x + x^2 + \frac{1}{3}\,x^3 + \frac{1}{4}\,x^4 + \cdots \tag{13}$$

Grading Scheme:

- *Procedure (10 pts):* $y = \sum a_n\,x^n$ *(2); Substitute into equation (1); Recurrence relation (2); How to find* a_0 *and* a_1 *(2); Get* a_n *one by one (2); Know what final answer should look like (2).*
- *Details (20 pts):* y'' *(2);* y' *(2); Equation (2);* $2\,a_2 - a_0 = 0$ *(2);* $(n+2)\,a_{n+2} - a_n = 0$ *(2);* $a_0 = 2$ *(2);* $a_1 = 1$ *(2);* a_2 — a_4 *(1+1+1);* $y = \cdots$ *(3).*
- *Common mistakes:*
    - *Don't know how to obtain* a_0, a_1.
    - *Write* $y = 2 + x + x^2 + \frac{1}{3}\,x^3 + \frac{1}{4}\,x^4$ *without the "⋯". (Didn't deduct any point this time. But* ***will take point(s) off in the final****).*
    - $y = a_0 + a_1\,x + \frac{a_2}{2!}\,x^2 + \cdots$ *after finding* a_n.

Problem 2 (20 pts) Find the general solution of

$$x^2\, y'' + 2\, x\, y' + y = 0. \tag{14}$$

Solution.

Set $y = x^r$ we reach the indicial equation

$$r\,(r-1) + 2\,r + 1 = 0 \Longrightarrow r^2 + r + 1 = 0 \Longrightarrow r_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i. \tag{15}$$

So the general solution is

$$y = C_1\, x^{-1/2} \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + C_2\, x^{-1/2} \sin\left(\frac{\sqrt{3}}{2} \ln x\right). \tag{16}$$

Grading Scheme:

- *Procedure (8 pts): Recognize it's Euler equation and know how to use characteristic/indicial equation (4); Know what the solution should look like (4).*
- *Details (12 pts): Correct characteristic/indicial equation (4); Correct roots (2+2); Correct solution (4).*
- *Common mistakes:*
  - *Didn't recognize the type of the equation.*
  - *All kinds of wrong form of solutions:* $e^{-1/2}$ $\cos\cdots$, $e^{-x/2}$ $\cos$ $\frac{\sqrt{3}}{2}$ $x\cdots$, $e^{-x/2}$ $\cos$ $\frac{\sqrt{3}}{2}$ $\ln$ $x\cdots$, $e^{-x/2} \cos\left(\ln \frac{\sqrt{3}}{2}\, x\right)\ldots$. *All possible combinations.*

Problem 3 (15 pts) Determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the differential equation

$$(1+x^3)\, y'' + 4\, x\, y' + 4\, y = 0; \qquad x_0 = 0,\ x_0 = 2. \tag{17}$$

Solution. Write the equation into standard form

$$y'' + \frac{4\, x}{1+x^3}\, y' + \frac{4}{1+x^3}\, y = 0. \tag{18}$$

We see that the singular points are solutions to

$$x^3 + 1 = 0. \tag{19}$$

or equivalently

$$x^3 = -1. \tag{20}$$

To find all such x, we need to write $-1 = R\, e^{i\theta}$. Clearly $R = 1$. To determine θ we solve

$$\cos\theta = -1, \qquad \sin\theta = 0 \tag{21}$$

which gives $\theta = \pi + 2\, k\, \pi$. Thus the solutions are given by

$$x = e^{i\frac{2k+1}{3}\pi}. \tag{22}$$

Notice that k and $k+3$ gives the same x. Therefore the three roots are given by setting $k = 0, 1, 2$.

$$k = 0 \Longrightarrow x = e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}\, i;\ \ k = 1 \Longrightarrow x = -1;\ \ k = 2 \Longrightarrow x = \frac{1}{2} - \frac{\sqrt{3}}{2}\, i. \tag{23}$$

Now we discuss

- $x_0 = 0$. The distance from 4 to the three roots are:

$$\left|0 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\, i\right)\right| = 1 \tag{24}$$

$$|0 - (-1)| = 1; \tag{25}$$

$$\left|0 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\, i\right)\right| = 1; \tag{26}$$

  The smallest distance is 1. So the radius of convergence is at least 1.

- $x_0 = 2$. The distances are

$$\begin{aligned} \left|2 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\, i\right)\right| &= \left|\frac{3}{2} - \frac{\sqrt{3}}{2}\, i\right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}; && (27)\\ |2 - (-1)| &= 3; && (28)\\ \left|2 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\, i\right)\right| &= \sqrt{3}. && (29) \end{aligned}$$

  The smallest distance is $\sqrt{3}$. So the radius of convergence is $\sqrt{3}$.

Grading Scheme:

- *Procedure (5 pts): Know need to find singular points (1); Know need to compute distances (1); Know need to find the shortest one (1); Know all the above should be carried out in the complex plane (2).*
- *Details (10 pts):* $1 + x^3 = 0$ *(1); Correct solutions (1+1+1); Discussion at* $x_0 = 0$ *(3); Discussion at* $x_0 = 2$ *(3);*
- *Remarks:*
  - *Trying to solve it (3) and apply ratio test (2) gets 5 pts (qualify as "know the procedure"). The other 10 pts for successfully carry this plan out.*

Problem 4 (15 pts) Find the general solution:

$$y'' + y = \frac{1}{\sin t}. \tag{30}$$

Solution.

- This problem should be solved using variation of parameters.
- First solve the homogeneous equation $y'' + y = 0$:

$$y_1 = \cos t, \qquad y_2 = \sin t. \tag{31}$$

- Compute the Wronskian:

$$W[y_1, y_2] = y_1\, y_2' - y_1'\, y_2 = 1. \tag{32}$$

- Compute the integrals:

$$\begin{aligned} \int \frac{-g\, y_2}{W} &= \int \frac{-\frac{1}{\sin t} \sin t}{1}\, \mathrm{d}t \\ &= \int -1\, \mathrm{d}t \\ &= -t; \end{aligned} \tag{33}$$

$$\begin{aligned} \int \frac{g\, y_1}{W} &= \int \frac{\frac{1}{\sin t} \cos t}{1}\, \mathrm{d}t \\ &= \int \frac{\cos t}{\sin t}\, \mathrm{d}t \\ &= \int \frac{\mathrm{d}(\sin t)}{\sin t} \\ &= \ln|\sin t|. \end{aligned} \tag{34}$$

- Get y_p.

$$y_p = t\, y_1 + \ln|\sin t|\, y_2 = -t \cos t + (\ln|\sin t|) \sin t. \tag{35}$$

- General solution is

$$y = C_1 \cos t + C_2 \sin t - t \cos t + (\ln|\sin t|) \sin t. \tag{36}$$

Grading Scheme:

- *Procedure (5 pts): Know to use variation of parameters (1); Know need to solve homogeneous equation first (1); Know the formulas (1); Know $y_1 = u_1\, y_1 + u_2\, y_2$ (1); Know $y = C_1\, y_1 + C_2\, y_2 + y_p$ (1).*
- *Details (10 pts): Solution to homogeneous equation (2); correct g (1); Wronskian (1); u_1 (2); u_2 (2); y_p (1); y (1).*
- *Common mistake:*
    - *Didn't know how to integrate $\frac{\cos t}{\sin t}$.*

Problem 5 (15 pts) Find the general solution for

$$y^{(5)} - 3\,y^{(4)} + 3\,y''' - 3\,y'' + 2\,y' = 0. \tag{37}$$

Solution. Characteristic equation:

$$r^5 - 3\,r^4 + 3\,r^3 - 3\,r^2 + 2\,r = 0. \tag{38}$$

Easy to see $r_1 = 0$. The other four solutions would come from

$$r^4 - 3\,r^3 + 3\,r^2 - 3\,r + 2 = 0. \tag{39}$$

Observe that $r_2 = 1$ is the 2nd root. Factorize

$$r^4 - 3\,r^3 + 3\,r^2 - 3\,r + 2 = (r-1)\,(r^3 - 2\,r^2 + r - 2). \tag{40}$$

We need to solve

$$r^3 - 2\,r^2 + r - 2 = 0 \tag{41}$$

which gives $r_3 = 2$. As

$$r^3 - 2\,r^2 + r - 2 = (r-2)\,(r^2 + 1) \tag{42}$$

the last two roots are $r_{4,5} = \pm i$.

So the general solution is

$$y = C_1 + C_2\,e^t + C_3\,e^{2t} + C_4 \cos t + C_5 \sin t. \tag{43}$$

Grading Scheme:

- *As in midterm 1, no "procedure" points for advanced and challenge problems.*
- *Characteristic equation (2); Roots (2×5); Solution (3).*
- *Common mistakes:*
  - *Didn't solve the characteristic equation correctly.*

Problem 6 (5 pts) Consider the equation

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{44}$$

Assume that 0 is a singular point. Let $\{y_1, y_2\}$ be a pair of solutions. Prove: If there are real, non-integer, numbers $r_1 \neq r_2$ such that both $Y_1 := x^{r_1}\, y_1$ and $Y_2 := x^{r_2}\, y_2$ are analytic at 0 with $Y_1(0) \neq 0, Y_2(0) \neq 0$, then 0 is regular singular.

Proof. Since y_1, y_2 are solutions,

$$y_1'' + p(x)\, y_1' + q(x) y_1 = 0, \qquad y_2'' + p(x)\, y_2' + q(x) y_2 = 0. \tag{45}$$

Treating this as a system with unknown p, q we reach

$$y_1'\, p + y_1\, q \;=\; -y_1'' \tag{46}$$

$$y_2'\, p + y_2\, q \;=\; -y_2'' \tag{47}$$

This gives

$$p = \frac{-y_1''\, y_2 + y_2''\, y_1}{y_2\, y_1' - y_1\, y_2'}; \qquad q = \frac{-y_1''\, y_2' + y_2''\, y_1'}{y_1\, y_2' - y_2\, y_1'}. \tag{48}$$

Now recall $y_1 = x^{-r_1}\, Y_1$, $y_2 = x^{-r_2}\, Y_2$. Then we have

$$y_1' \;=\; -r_1\, x^{-r_1-1}\, Y_1 + x^{-r_1}\, Y_1' \tag{49}$$

$$y_2' \;=\; -r_2\, x^{-r_2-1}\, Y_2 + x^{-r_2}\, Y_2' \tag{50}$$

$$y_1'' \;=\; r_1\, (r_1+1) x^{-r_1-2}\, Y_1 - 2\, r_1\, x^{-r_1-1}\, Y_1' + x^{-r_1}\, Y_1'' \tag{51}$$

$$y_2'' \;=\; r_2\, (r_2+1)\, x^{-r_2-2}\, Y_2 - 2\, r_2\, x^{-r_2-1}\, Y_2' + x^{-r_2}\, Y_2''. \tag{52}$$

Substitute into the formula for p and simplify, we get

$$x\, p(x) = \frac{[r_2\, (r_2+1) - r_1\, (r_1+1)]\, Y_1\, Y_2 + 2\, x\, [r_1\, Y_1'\, Y_2 - r_2\, Y_2'\, Y_1] - x^2\, [Y_1''\, Y_2 - Y_2''\, Y_1]}{(r_2 - r_1)\, Y_1\, Y_2 + x\, (Y_1'\, Y_2 - Y_2'\, Y_1)}. \tag{53}$$

As Y_1, Y_2 are analytic at 0, so are all their derivatives and therefore both the numerator and the denominator are analytic at 0. All we need to check is whether the denominator is 0 at 0.

Setting $x = 0$ in the denominator becomes $(r_2 - r_1)\; Y_1(0)\; Y_2\,(0)$ which is not 0 according to the assumptions in the problem ($r_1 \neq r_2, Y_1(0) \neq 0, Y_2(0) \neq 0$). Therefore $x\, p$ is analytic at 0.

That $x^2\, q$ is analytic at 0 can be shown similarly. □

Grading Scheme:

- *Know what to do (3); Discussion of $x\, p$ (1); Discussion of $x^2\, q$ (1).*
- *Common mistakes:*
  - *Didn't take a good look at homework 8 solution.*
  - *Use Fuchs' Theorem to prove the claim. This problem is a "weaker version" of Fuchs' Theorem so such "proof" is essentially cyclic. It's like proving "There is no integer solution to $x^3 + y^3 = z^3$" with "This is true because of Fermat's Last Theorem".*