Lecture 35 Matrix Exponentials

12/05/2011

What really happens when we have n linearly independent eigenvectors.

- Recall that when we have n linearly independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, then the general solution is given by

$$C_1 e^{\lambda_1 t} \boldsymbol{x}_1 + \cdots + C_n e^{\lambda_n t} \boldsymbol{x}_n. \tag{1}$$

- Now consider the initial value problem: What are $C_1, \ldots, C_n$ after all? Setting $t = 0$ we obtain

$$\boldsymbol{x}(0) = C_1 \boldsymbol{x}_1 + \cdots + C_n \boldsymbol{x}_n. \tag{2}$$

  We can put $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ together to form a matrix:

$$X := \begin{pmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{pmatrix}. \tag{3}$$

  Now we reach

$$\boldsymbol{x}(0) = \begin{pmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = X \boldsymbol{c} \tag{4}$$

  where the vector $\boldsymbol{c} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$.

- Next we try to write the general solution into matrix form.

$$C_1 e^{\lambda_1 t} \boldsymbol{x}_1 + \cdots + C_n e^{\lambda_n t} \boldsymbol{x}_n = \begin{pmatrix} e^{\lambda_1 t} \boldsymbol{x}_1 & \cdots & e^{\lambda_n t} \boldsymbol{x}_n \end{pmatrix} \boldsymbol{c} = X \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} \boldsymbol{c}. \tag{5}$$

  Now as the matrix X is nonsingular (because the $\boldsymbol{x}_i$'s are linearly independent), we have

$$\boldsymbol{x}(0) = X \boldsymbol{c} \Longleftrightarrow \boldsymbol{c} = X^{-1} \boldsymbol{x}(0). \tag{6}$$

  Putting the above together, we reach

$$\boldsymbol{x}(t) = X \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} X^{-1} \boldsymbol{x}(0). \tag{7}$$

- Now we see that the matrix

$$\Phi(t) := X \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} X^{-1} \tag{8}$$

  is a significant object: It gives a universal formula for all solutions:

$$\boldsymbol{x}(t) = \Phi(t) \boldsymbol{x}(0). \tag{9}$$

- We now try to find the relation between $\Phi(t)$ and A. Since $\Phi(t)$ is of the form $X \cdot \text{something} \cdot X^{-1}$, we explore what happens if we try to write A in a similar way.

  Recall that each $\boldsymbol{x}_i$ is an eigenvector corresponding to eigenvalue λ_i. That is

$$A \boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i. \tag{10}$$

  Putting all $\boldsymbol{x}_i$'s in a row to form the matrix X, we get

$$A X = A \begin{pmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \boldsymbol{x}_1 & \cdots & \lambda_n \boldsymbol{x}_n \end{pmatrix} = X \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{11}$$

Multiply both sides by X^{-1} **from the right** – recall that matrix multiplication is not commutative – we reach

$$A = X \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} X^{-1}. \tag{12}$$

- Comparing

$$A = X \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} X^{-1} \tag{13}$$

with

$$\Phi(t) := X \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} X^{-1} \tag{14}$$

we want to say

$$\Phi(t) = e^{A t}. \tag{15}$$

Then the solution to

$$\dot{\boldsymbol{x}} = A \boldsymbol{x} \tag{16}$$

is simply

$$\boldsymbol{x}(t) = e^{A t} \boldsymbol{x}(0), \tag{17}$$

a *perfect generalization* of the single linear equation:[1]

$$\dot{x} = a x \Longrightarrow x(t) = e^{a t} x(0). \tag{18}$$

Definition of matrix exponentials.

- However how to define e^A for a general matrix A?
- Matrix functions: Given a square matrix A, what kind of functions can be readily generalized to take A as its variable? Polynomials – as matrix products are already well-defined. For example

$$f(x) = x^3 + 3x - 1 \Longrightarrow f(A) = A^3 + 3A - I. \tag{19}$$

- Now how to define e^A? Taylor expansion!

$$e^x = 1 + x + \frac{x^2}{2} + \cdots \Longrightarrow e^A := I + A + \frac{A^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{20}$$

- Is this what we want?
  - Check the special case:

$$A = X \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} X^{-1} \Longrightarrow$$

$$\begin{aligned} (A t)^k &= \left(X \begin{pmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{pmatrix} X^{-1} \right) \cdots \left(X \begin{pmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{pmatrix} X^{-1} \right) \\ &= X \begin{pmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{pmatrix} X^{-1} X \begin{pmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{pmatrix} X^{-1} \cdots X^{-1}. \end{aligned} \tag{21}$$

1. However see homework: Such generalizations are actually subtle.

Recall that matrix multiplication is associative, which means we can freely "pair up" adjacent matrices:

$$\begin{aligned}(A\,t)^k &= X\begin{pmatrix}\lambda_1 t & & 0\\ & \ddots & \\ 0 & & \lambda_n t\end{pmatrix}(X^{-1}X)\begin{pmatrix}\lambda_1 t & & 0\\ & \ddots & \\ 0 & & \lambda_n t\end{pmatrix}(X^{-1}X)\cdots X^{-1}\\ &= X\begin{pmatrix}\lambda_1 t & & 0\\ & \ddots & \\ 0 & & \lambda_n t\end{pmatrix}\cdots\begin{pmatrix}\lambda_1 t & & 0\\ & \ddots & \\ 0 & & \lambda_n t\end{pmatrix}X^{-1}\\ &= X\begin{pmatrix}\lambda_1^k t^k & & 0\\ & \ddots & \\ 0 & & \lambda_n^k t^k\end{pmatrix}X^{-1}.\end{aligned}$$

Now it's easy to see

$$e^{At} = X\begin{pmatrix}\sum \frac{\lambda_1^k t^k}{k!} & & \\ & \ddots & \\ & & \sum \frac{\lambda_n^k t^k}{k!}\end{pmatrix}X^{-1} = X\begin{pmatrix}e^{\lambda_1 t} & & 0\\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{pmatrix}X^{-1} = \Phi(t). \tag{22}$$

Matrix exponentials and first order systems.

Theorem 1. *Consider the first order system $\dot{\boldsymbol{x}} = A\,\boldsymbol{x}$. Then $\Phi(t) = e^{At}$ as defined above satisfies*

$$\dot{\Phi}(t) = A\,\Phi(t), \qquad \Phi(0) = I. \tag{23}$$

and consequently the solution of

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x}, \qquad \boldsymbol{x} = \boldsymbol{x}(0) \text{ at } t = 0. \tag{24}$$

is given by

$$\boldsymbol{x}(t) = \Phi(t)\,\boldsymbol{x}(0). \tag{25}$$

Proof. $\Phi(0) = X^{-1}\,I\,X = I$. *Compute*

$$\dot{\Phi}(t) = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A^k t^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{A^{k+1} t^k}{k!} = \sum_{k=0}^{\infty} A\,\frac{A^k t^k}{k!} = A\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = A\,\Phi(t). \tag{26}$$

The last few stpes may seem too obvious to worth writing down, but in fact it's important to clearly write down every "obvious" step. See homework.

Now we have

$$\dot{\boldsymbol{x}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)\,\boldsymbol{x}(0)) = \dot{\Phi}(t)\,\boldsymbol{x}(0) = A\,\Phi(t)\,\boldsymbol{x}(0) = A\,\boldsymbol{x}(t). \tag{27}$$

Finally ($\boldsymbol{x}$ *at* $t = 0) = \Phi(0)\,\boldsymbol{x}(0) = I\,\boldsymbol{x}(0) = \boldsymbol{x}(0)$. □

Remark 2. Note that in the above proof what we actually show is that $\Phi(t)\,\boldsymbol{x}(0)$ is **a** solution of the system. That this suffices is due to the fact that the solution is unique – so "**a** solution" gets a "free upgrade" to "**the** solution".

Calculation of matrix exponentials – Simple case.

- Clearly it's not a good idea to use the definition:

$$e^A := I + A + \frac{A^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{28}$$

- When A has n linearly independent eigenvectors, we have shown that

$$A = X\,\Lambda\,X^{-1} \tag{29}$$

where $X = \begin{pmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_n \end{pmatrix}$ is the matrix formed by putting these n eigenvectors in a row, and $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ is a diagonal matrix with the corresponding eigenvalues as diagonal entries. In this case we know that

$$e^A = X \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} X^{-1}. \tag{30}$$

Example 3. Compute e^A with

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}. \tag{31}$$

Solution. First obtain the eigenvalues:

$$\det \begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda_{1,2} = 1, -1. \tag{32}$$

Next find a set of 2 linearly independent eigenvectors:

- For 1, solve

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{33}$$

- for -1, solve

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \tag{34}$$

So

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \tag{35}$$

and

$$A = X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X^{-1} \Longrightarrow e^A = X \begin{pmatrix} e^1 & 0 \\ 0 & e^{-1} \end{pmatrix} X^{-1}. \tag{36}$$

To get the final answer we need to find X^{-1}, through solving $X\,X^{-1} = I$ using Gaussian elimination:

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \end{aligned} \tag{37}$$

We get

$$X^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{38}$$

Now we compute

$$e^A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}e - \frac{1}{2}e^{-1} & -\frac{1}{2}e + \frac{1}{2}e^{-1} \\ \frac{3}{2}e - \frac{3}{2}e^{-1} & -\frac{1}{2}e + \frac{3}{2}e^{-1} \end{pmatrix}. \tag{39}$$

Calculation of matrix exponentials – General case.

- What if we do not have n linearly independent eigenvectors? **Note**:

$$A = X \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} X^{-1} \Longrightarrow \text{Each column of } X \text{ is an eigenvector} \tag{40}$$

  Therefore when we do not have n linearly independent eigenvectors, it's not possible to reduce A to a diagonal matrix – that is not possible to "diagonalize" A.

- Key property: If $A = X\,B\,X^{-1}$, then $e^A = X\,e^B\,X^{-1}$.
- Question: What is the simplest matrix that all $n \times n$ matrices A can be reduced to?
- Answer: Jordan canonical form.

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} \tag{41}$$

  where each $J_k = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$ is called a "Jordan block".

  Theorem 4. *Any $n \times n$ matrix can be written as $A = X\,J\,X^{-1}$ where J is of the above form. Furthermore, the columns of X (denote by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$) corresponding to one "Jordan block" is related in the following manner:*

$$(A - \lambda I)\,\boldsymbol{x}_1 = 0; \quad (A - \lambda I)\,\boldsymbol{x}_{i+1} = \boldsymbol{x}_i. \tag{42}$$

  It may help to see an example. Suppose we have

$$A = X \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} X^{-1}. \tag{43}$$

  Multiply both sides by X from right, we reach

$$A\,(\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,) = A\,X = X \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} = (\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,) \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}. \tag{44}$$

  Carry out the multiplication we reach

$$(\,A\,\boldsymbol{x}_1 \;\; A\,\boldsymbol{x}_2 \;\; A\,\boldsymbol{x}_3\,) = (\,\lambda\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_1 + \lambda\,\boldsymbol{x}_2 \;\; \boldsymbol{x}_2 + \lambda\,\boldsymbol{x}_3\,) \tag{45}$$

  which means

$$(A - \lambda)\,\boldsymbol{x}_1 = 0 \tag{46}$$
$$(A - \lambda)\,\boldsymbol{x}_2 = \boldsymbol{x}_1 \tag{47}$$
$$(A - \lambda)\,\boldsymbol{x}_3 = \boldsymbol{x}_2. \tag{48}$$

- How to compute e^J.
  - Observation I:

$$\exp \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} = \begin{pmatrix} e^{J_1} & & \\ & \ddots & \\ & & e^{J_k} \end{pmatrix}. \tag{49}$$

  - Observation II:

$$e^{\lambda I + A} = e^{\lambda I}\,e^A. \tag{50}$$

for any matrix A.

- Observation III: Let $B=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$ be $k\times k$, then

$$B^2=\begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, \qquad B^3=\begin{pmatrix} 0 & 0 & 0 & 1 & \\ & 0 & 0 & 0 & \ddots \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, \ldots \tag{51}$$

consequently

$$B^k=0, \tag{52}$$

and

$$e^B=\begin{pmatrix} 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{(k-1)!} \\ & 1 & 1 & & \vdots \\ & & \ddots & \ddots & \frac{1}{2} \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \tag{53}$$

and

$$e^{Bt}=\begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ & & & 1 & t \\ & & & & 1 \end{pmatrix}. \tag{54}$$

Example 5. Solve

$$\dot{\boldsymbol{x}}=\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\boldsymbol{x} \tag{55}$$

using matrix exponentials.
Solution. The matrix is already in Jordan canonical form. We see that there are two Jordan blocks:

$$A=\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \qquad J_1=\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad J_2=(\ 1\ ). \tag{56}$$

By Observation I we have

$$e^{At}=\begin{pmatrix} e^{J_1 t} & 0 \\ 0 & e^{J_2 t} \end{pmatrix}. \tag{57}$$

Clearly $e^{J_2 t}=(\ e^t\ )$. For $e^{J_1 t}$ we use the next two observations:

$$\begin{aligned} e^{J_1 t} &= e^{3It+\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}t} \\ &= e^{3It}\, e^{\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}t} \\ &= e^{3t}\, I\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{3t} & t\,e^{3t} & \frac{t^2\,e^{3t}}{2} \\ 0 & e^{3t} & t\,e^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}. \end{aligned} \tag{58}$$

Therefore

$$e^{At} = \begin{pmatrix} e^{3t} & t\,e^{3t} & \frac{t^2\,e^{3t}}{2} & 0 \\ 0 & e^{3t} & t\,e^{3t} & 0 \\ 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}. \tag{59}$$

The general solution is now

$$e^{At}\,\boldsymbol{c} = c_1 \begin{pmatrix} e^{3t} \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t\,e^{3t} \\ e^{3t} \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \frac{t^2\,e^{3t}}{2} \\ t\,e^{3t} \\ e^{3t} \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^t \end{pmatrix}. \tag{60}$$

Remark. Now we see where the t, t^2, ... etc. come from! And furthermore we see why how many powers of t are needed cannot be determined by the algebraic and geometric multiplicities alone: Compute the following two A's (in the context of computing e^{At}):

$$\begin{pmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & \\ & & & 3 \end{pmatrix}. \tag{61}$$

In both cases, the eigenvalue 3 has algebraic multiplicity 4 and geometric multiplicity 2. However in the former case e^{At} involves only e^{3t} and $t\,e^{3t}$, while in the latter case $t^2\,e^{3t}$ will also appear.