Lecture 34 Solving First Order Homogeneous Constant Coefficient System (Cont.)

12/02/2011

Review.

- Consider $\dot{\boldsymbol{x}} = A\,\boldsymbol{x}$.
- If we can find n linearly independent eigenvectors $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ (note that some of the λ_i's may repeat), then the general solution is given by

$$C_1\, e^{\lambda_1 t}\, \boldsymbol{x}_0^{(1)} + \cdots + C_n\, e^{\lambda_n t}\, \boldsymbol{x}_0^{(n)}. \tag{1}$$

What if we don't have enough eigenvectors.

- How many are missing: Algebraic and geometric multiplicities.
  - Algebraic multiplicity: How many times an eigenvalue is repeated as a root of the polynomial

$$\det\,(A - \lambda\, I) = 0. \tag{2}$$

    For example, let $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Then $\det\,(A - \lambda\, I) = (1 - \lambda)^3\,(2 - \lambda)$ which means there are two eigenvalues: 1 and 2. The eigenvalue 1 has algebraic multiplicity 3 while the eigenvalue 2 has algebraic multiplicity 1.

  - Geometric multiplicity: Given an eigenvalue, how many linearly independent eigenvectors (corrsponding to that particular eigenvalue) are there.

    For the above example, the geometric multiplicity for the eigenvalue 2 is clearly 1, while the geometric multiplicity for the eigenvalue 1 is only 1, not 3. To see this, note that

$$(A - 1 \cdot I)\,\boldsymbol{x} = \boldsymbol{0} \Longleftrightarrow \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

    which gives $x_2 = 0, x_3 = 0, x_4 = 0$. So all the eigenvectors corresponding to 1 are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

  - We have the following:

    Theorem. *Let A be any $n \times n$ matrix and λ be one of its eigenvalues. Then*

    - *The geometric multiplicity of $\lambda \leqslant$ The algebraic multiplicity of λ;*
    - *The geometric multiplicity of λ is at least 1.*

    Corollary. *Following the theorem, we can conclude*

    - *The sum of geometric multiplicities of all eigenvalues of A is at most n;*
    - *When there are n distinct eigenvalues, the sum of all geometric multiplicities is exactly n.*

  - What the above mean to us:
    - When there are n distinct eigenvalues, we can always find n linearly independent eigenvectors $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$, and the general solution is

$$C_1\, e^{\lambda_1 t}\, \boldsymbol{x}_0^{(1)} + \cdots + C_n\, e^{\lambda_n t}\, \boldsymbol{x}_0^{(n)}. \tag{5}$$

 - When some eigenvalues are repeated, we may or may not be able to find n linearly independent eigenvectors.

 - Suppose we only have $k<n$ linearly independent eigenvectors, the general solution becomes

$$C_1\,e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)}+\cdots+C_k\,e^{\lambda_k t}\,\boldsymbol{x}_0^{(k)}+C_{k+1}\,\boldsymbol{x}^{(k+1)}(t)+\cdots+C_n\,\boldsymbol{x}^{(n)}(t). \tag{6}$$

 - **Question:** How to find $\boldsymbol{x}^{(k+1)}(t),\ldots,\boldsymbol{x}^{(n)}(t)$?

- Formulas for the simplest case.

 - Let λ be an eigenvalue with algebraic multiplicity 2 while geometric multiplicity 1. Let $\boldsymbol{x}_0$ be one eigenvector. Thus $e^{\lambda t}\,\boldsymbol{x}_0$ is a solution to the system. Our task is to find a second solution.

 - Try $e^{\lambda t}\,\boldsymbol{\xi}+t\,e^{\lambda t}\,\boldsymbol{\eta}$. Here $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two vectors that we need to find. Substitute into the equation we get

$$\lambda\,e^{\lambda t}\,\boldsymbol{\xi}+e^{\lambda t}\,\boldsymbol{\eta}+\lambda\,t\,e^{\lambda t}\,\boldsymbol{\eta}=e^{\lambda t}\,A\,\boldsymbol{\xi}+t\,e^{\lambda t}\,A\,\boldsymbol{\eta}. \tag{7}$$

 Collecting similar terms together, and cancel the factor $e^{\lambda t}$, we reach

$$[(A-\lambda\,I)\,\boldsymbol{\xi}-\boldsymbol{\eta}]+t\,[A\,\boldsymbol{\eta}-\lambda\,\boldsymbol{\eta}]=\mathbf{0}. \tag{8}$$

 Thus $e^{\lambda t}\,\boldsymbol{\xi}+t\,e^{\lambda t}\,\boldsymbol{\eta}$ solves the equation if and only if

$$\begin{aligned}(A-\lambda\,I)\,\boldsymbol{\xi} &= \boldsymbol{\eta} && (9)\\ (A-\lambda\,I)\,\boldsymbol{\eta} &= \mathbf{0}. && (10)\end{aligned}$$

 - Thus we see that we can take $\boldsymbol{\eta}=\boldsymbol{x}_0$ and then solve

$$(A-\lambda\,I)\boldsymbol{\xi}=\boldsymbol{x}_0 \tag{11}$$

 to get $\boldsymbol{\xi}$.

 - Note that such $\boldsymbol{\xi}$ is clearly not unique, since if $\boldsymbol{\xi}$ is a solution, then the sum $\boldsymbol{\xi}+c\,\boldsymbol{x}_0$ for any constant c is also a solution.

 - We only need one such $\boldsymbol{\xi}$.

 - This is guaranteed to work:

 Theorem. *For λ and $\boldsymbol{x}_0$ as in the above, such $\boldsymbol{\xi}$ always exists, and is unique (upto $+c\,\boldsymbol{x}_0$)*

 Proof. *Unfortunately I couldn't figure out a simple proof even for this simplest case.* □

 - Such $\boldsymbol{\xi}$ is called "*generalized eigenvectors*".

Example. Solve

$$\boldsymbol{x}'=\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\boldsymbol{x}. \tag{12}$$

Solution. First find eigenvalues:

$$\begin{aligned}0=\det\begin{pmatrix}1-\lambda & 1 & 1\\ 2 & 1-\lambda & -1\\ 0 & -1 & 1-\lambda\end{pmatrix} &= (1-\lambda)^3-2-(1-\lambda)-2\,(1-\lambda)\\ &= -\lambda^3+3\,\lambda^2-4\\ &= -(\lambda+1)\,(\lambda^2-4\,\lambda+4)\\ &= -(\lambda+1)\,(\lambda-2)^2. && (13)\end{aligned}$$

We have two eigenvalues, -1 and 2.

Now find eigenvectors.

- Eigenvectors for -1: Solve

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14}$$

We have

$$\begin{aligned}\begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} &\longrightarrow \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

So the eigenvectors are characterized by

$$\begin{array}{rcl} 2x_1+x_2+x_3 &=& 0 \\ x_2-2x_3 &=& 0 \end{array} \Longleftrightarrow \begin{array}{rcl} x_1 &=& -\dfrac{3}{2}x_3 \\ x_2 &=& 2x_3 \end{array} \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=x_3\begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix}. \tag{16}$$

The first solution in our set of fundamental solutions is thus

$$e^{-t}\begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix}. \tag{17}$$

- Eigenvectors for 2: Solve

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

We have

$$\begin{aligned}\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} &\longrightarrow \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{19}$$

Thus the eigenvectors are given by

$$\begin{array}{rcl} x_1-x_2-x_3 &=& 0 \\ x_2+x_3 &=& 0 \end{array} \Longleftrightarrow \begin{array}{rcl} x_1 &=& 0 \\ x_2 &=& -x_3 \end{array} \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=x_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{20}$$

As 2 has algebraic multiplicity 2, we need to find its generalized eigenvectors. We thus obtained our second solution in the set of fundamental solutions:

$$e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{21}$$

- Generalized eigenvectors for 2: Solve

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{22}$$

We have

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

So the generalized eigenvectors are characterized by

$$y_1 - y_2 - y_3 = 0 \tag{24}$$

$$y_2 + y_3 = 1. \tag{25}$$

Keeping in mind that all we need is one such vectors, we take

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{26}$$

The third solution in the set of fundamental solutions is thus

$$e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\, e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{27}$$

The general solution is now given by

$$C_1 e^{-t}\begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3\left[e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\, e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right]. \tag{28}$$

- What happens in the general case
  - In general, let λ be an eigenvalue with algebraic multiplicity m and geometric multiplicity k. Then we may[1] need to consider solutions of the form

    $$e^{\lambda t}\boldsymbol{\xi}_0 + t\, e^{\lambda t}\boldsymbol{\xi}_1 + \cdots + t^{m-k} e^{\lambda t}\boldsymbol{\xi}_{m-k}. \tag{29}$$

    Here $\boldsymbol{\xi}_0$ is an eigenvector, while $\boldsymbol{\xi}_i$'s are decided successively through

    $$(A - \lambda I)\,\boldsymbol{\xi}_{i+1} = \boldsymbol{\xi}_{\boldsymbol{i}} \tag{30}$$

    The tricky issue here is that the eigenvector $\boldsymbol{\xi}_0$ cannot be decided a priori.
  - Some understanding of the above subtleties as well as true understanding of the whole solution procedure of 1st order constant coefficient systems can be obtained through the next lecture.

1. Whether we really need to go up to t^{m-k} cannot be determined by knowledge of only m and k, as it depends on the detailed structure of the matrix, or more specifically, depends on what the Jordan canonical form of the matrix looks like. See next lecture for more details.