Lecture 33 Solving First Order Homogeneous Constant Coefficient System

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Idea.

- Need to solve

$$\begin{aligned} \dot{x}_1 &= a_{11}\,x_1 + \cdots + a_{1n}\,x_n \quad (1)\\ \vdots \; &\;\vdots\; \vdots \\ \dot{x}_n &= a_{n1}\,x_1 + \cdots + a_{nn}\,x_n \quad (2) \end{aligned}$$

  or in matrix form:

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x} \tag{3}$$

  with

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}. \tag{4}$$

- What do we know about the solution:
  - General solution is of the form

$$C_1\,\boldsymbol{x}^{(1)} + \cdots + C_n\,\boldsymbol{x}^{(n)} \tag{5}$$

    with $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ solutions, linearly independent.
  - Therefore, all we need to do is to find n linearly independent solutions.
- Try $e^{\lambda t}\,\boldsymbol{x}_0$ with $\boldsymbol{x}_0$ a constant vector. Compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t}\,\boldsymbol{x}_0) = \lambda\,e^{\lambda t}\,\boldsymbol{x}_0 \tag{6}$$

  The equation now becomes

$$\lambda\,e^{\lambda t}\,\boldsymbol{x}_0 = A\,e^{\lambda t}\,\boldsymbol{x}_0 \Longleftrightarrow A\,\boldsymbol{x}_0 = \lambda\,\boldsymbol{x}_0 = \lambda\,I\,\boldsymbol{x}_0 \Longleftrightarrow (A - \lambda\,I)\,\boldsymbol{x}_0 = \boldsymbol{0}. \tag{7}$$

- Therefore: $e^{\lambda t}\,\boldsymbol{x}_0$ is a solution $\Longleftrightarrow$ λ is an eigenvalue and $\boldsymbol{x}_0$ is an corresponding eigenvector.
- How do we tell whether solutions $e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)}, \ldots, e^{\lambda_n t}\,\boldsymbol{x}_0^{(n)}$ are linearly independent or not?
  - They are linearly independent $\Longleftrightarrow$ their Wronskian is nonzero at $t = 0$ $\Longleftrightarrow$ $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$ are linearly independent.
- Conclusion: If we can find n linearly independent eigenvectors $\boldsymbol{x}_0^{(1)}$, ..., $\boldsymbol{x}_0^{(n)}$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ (note that some of the λ_i's may repeat), then the general solution is given by

$$C_1\,e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)} + \cdots + C_n\,e^{\lambda_n t}\,\boldsymbol{x}_0^{(n)}. \tag{8}$$

  Remark. The textbook, due to its intending to discuss the phase plane and the behavior as $t \nearrow \infty$ of the solutions, makes a distinction between n distinct eigenvalues and some eigenvalues are repeated. Since we focus on getting formulas for solutions, this distinction is not important anymore. There are only two cases: We have n linearly independent eigenvectors, or not. We deal with the former case in this lecture, and leave the latter to the next.

Examples.

Example 1. Solve

$$\begin{aligned} \dot{x}_1 &= x_1 + 4\,x_2 \quad (9)\\ \dot{x}_2 &= x_1 - 2\,x_2. \quad (10) \end{aligned}$$

Solution. First re-write into matrix form:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}. \tag{11}$$

Now we find the eigenvalues/eigenvectors for $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 + \lambda - 6 \Longrightarrow \lambda_1 = -3, \lambda_2 = 2. \tag{12}$$

Next find eigenvectors corresponding to -3: Solve

$$\begin{pmatrix} 1-(-3) & 4 \\ 1 & -2-(-3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow x_1 + x_2 = 0. \tag{13}$$

therefore the eigenvectors corresponding to -3 are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{14}$$

For 2 we have

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow x_1 = 4\, x_2 \tag{15}$$

so the eigenvectors corresponding to 2 are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \tag{16}$$

The general solution to the system is then given by

$$C_1\, e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\, e^{2t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{17}$$

or equivalently

$$x_1 = C_1\, e^{-3t} + 4\, C_2\, e^{2t}; \tag{18}$$
$$x_2 = -C_1\, e^{-3t} + C_2\, e^{2t}. \tag{19}$$

Example 2. Solve initial value problem (that is, find the **real** general solution)

$$\dot{x_1} = -x_1 + 5\, x_2; \qquad x_1(0) = 0 \tag{20}$$
$$\dot{x_2} = -4\, x_1 - 5\, x_2; \qquad x_2(0) = 1 \tag{21}$$

Solution. For initial value problems, we first find the general solution, then determine the constants using the initial conditions.

Preparation: Write the problem in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longrightarrow A = \begin{pmatrix} -1 & 5 \\ -4 & -5 \end{pmatrix}. \tag{22}$$

First find the eigenvalues:

$$\det \begin{pmatrix} -1-\lambda & 5 \\ -4 & -5-\lambda \end{pmatrix} = 0 \Longleftrightarrow \lambda^2 + 6\, \lambda + 25 = 0 \Longrightarrow \lambda_{1,2} = -3 \pm 4\, i. \tag{23}$$

Now find the corresponding eigenvectors:

- For the eigenvalue $-3 + 4\, i$, we solve

$$\begin{pmatrix} 2-4\, i & 5 \\ -4 & -2-4\, i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{24}$$

It may not be obvious at first sight that the two rows are linked by a constant factor, so we go through Guassian elimination:

$$\left(\begin{array}{ccc} 2-4\,i & 5 & 0\\ -4 & -2-4\,i & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc} 1 & \frac{1+2\,i}{2} & 0\\ -4 & -2-4\,i & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc} 1 & \frac{1+2\,i}{2} & 0\\ 0 & 0 & 0 \end{array}\right). \tag{25}$$

So $\left(\begin{smallmatrix} x_1\\ x_2 \end{smallmatrix}\right)$ is an eigenvector (corresponding to the eigenvalue $-3+4\,i$) if and only if

$$\left(\begin{array}{cc} 1 & \frac{1+2\,i}{2}\\ 0 & 0 \end{array}\right) \left(\begin{array}{c} x_1\\ x_2 \end{array}\right) = \left(\begin{array}{c} 0\\ 0 \end{array}\right) \Longleftrightarrow x_1 = \left(-\frac{1}{2} - i\right) x_2 \Longleftrightarrow \left(\begin{array}{c} x_1\\ x_2 \end{array}\right) = \left(\begin{array}{c} \left(-\frac{1}{2}-i\right) x_2\\ x_2 \end{array}\right) = x_2 \left(\begin{array}{c} -\frac{1}{2}-i\\ 1 \end{array}\right). \tag{26}$$

therefore the eigenvectors corresponding to $-3+4\,i$ are:

$$C_1 \left(\begin{array}{c} -\frac{1}{2}-i\\ 1 \end{array}\right) \tag{27}$$

with an arbitrary constant a.

- For $-3-4\,i$, similar calculation gives

$$C_2 \left(\begin{array}{c} -\frac{1}{2}+i\\ 1 \end{array}\right). \tag{28}$$

Thus the general solution is

$$\left(\begin{array}{c} x_1\\ x_2 \end{array}\right) = C_1\, e^{(-3+4i)t} \left(\begin{array}{c} -\frac{1}{2}-i\\ 1 \end{array}\right) + C_2\, e^{(-3-4i)} \left(\begin{array}{c} -\frac{1}{2}+i\\ 1 \end{array}\right). \tag{29}$$

However this is complex. How should we get the real solutions?

Notice that A is a real matrix. Therefore if $\boldsymbol{x} + i\,\boldsymbol{y}$ is a complex solution to the equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} + i\,\boldsymbol{y}) = A\,(\boldsymbol{x} + i\,\boldsymbol{y}) \Longleftrightarrow \dot{\boldsymbol{x}} + i\,\dot{\boldsymbol{y}} = A\,\boldsymbol{x} + i\,A\,\boldsymbol{y} \Longleftrightarrow \dot{\boldsymbol{x}} = A\,\boldsymbol{x} \text{ and } \dot{\boldsymbol{y}} = A\,\boldsymbol{y}. \tag{30}$$

Inspired by this, we look at the situation again. We know that $e^{(-3+4i)t}\left(\begin{smallmatrix} -\frac{1}{2}-i\\ 1 \end{smallmatrix}\right)$ solves the equation. Now expand

$$\begin{aligned} e^{(-3+4i)t} \left(\begin{array}{c} -\frac{1}{2}-i\\ 1 \end{array}\right) &= e^{-3t}\,[\cos 4\,t + i \sin 4\,t] \left[\left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right) + i \left(\begin{array}{c} -1\\ 0 \end{array}\right)\right] \\ &= e^{-3t} \left[\cos 4\,t \left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right) + \sin 4\,t \left(\begin{array}{c} 1\\ 0 \end{array}\right)\right] \\ &\quad + i\, e^{-3t} \left[\cos 4\,t \left(\begin{array}{c} -1\\ 0 \end{array}\right) + \sin 4\,t \left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right)\right]. \end{aligned} \tag{31}$$

Thus both

$$e^{-3t} \left[\cos 4\,t \left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right) + \sin 4\,t \left(\begin{array}{c} 1\\ 0 \end{array}\right)\right] \text{ and } e^{-3t} \left[\cos 4\,t \left(\begin{array}{c} -1\\ 0 \end{array}\right) + \sin 4\,t \left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right)\right] \tag{32}$$

are real solutions to the problem. And they are indeed guaranteed to be linearly independent[1]. Therefore the general (real) solution is

$$C_1\, e^{-3t} \left[\cos 4\,t \left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right) + \sin 4\,t \left(\begin{array}{c} 1\\ 0 \end{array}\right)\right] + C_2\, e^{-3t} \left[\cos 4\,t \left(\begin{array}{c} -1\\ 0 \end{array}\right) + \sin 4\,t \left(\begin{array}{c} -\frac{1}{2}\\ 1 \end{array}\right)\right] \tag{33}$$

1. See homework problem.

or in more detail:

$$x_1 = e^{-3t}\left[\left(-\frac{C_1}{2} - C_2\right)\cos 4\,t + \left(C_1 - \frac{C_2}{2}\right)\sin 4\,t\right] \tag{34}$$

$$x_2 = e^{-3t}\left[(C_1\cos 4\,t + C_2\sin 4\,t)\right]. \tag{35}$$

Finally we deal with the initial conditions: $x_1(0)=0$, $x_2(0)=1$ means

$$-\frac{C_1}{2} - C_2 = 0 \tag{36}$$

$$C_1 = 1 \tag{37}$$

So $C_1=1, C_2=-\frac{1}{2}$. The solution to the initial value problem is

$$x_1 = \frac{5}{4}e^{-3t}\sin 4\,t \tag{38}$$

$$x_2 = e^{-3t}\left[\cos 4\,t - \frac{1}{2}\sin 4\,t\right]. \tag{39}$$

Summary.

- To solve

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x} \tag{40}$$

1. Solve

$$\det(A - \lambda\,I) = 0 \tag{41}$$

to obtain all the eigenvalues;

2. For each eigenvalue, find all corresponding eigenvectors, represented as

$$a\,\boldsymbol{x}_1 + b\,\boldsymbol{x}_2 + \cdots \tag{42}$$

with $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ linearly independent.

3. If overall we have n eigenvectors already[2], then the general solution is

$$C_1\,e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)} + \cdots + C_n\,e^{\lambda_n t}\,\boldsymbol{x}_0^{(n)}. \tag{45}$$

where $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$ are the n eigenvectors, and $\lambda_1, \ldots, \lambda_n$ (may or may not be distinct) are the corresponding eigenvalues.

   - In the case of complex eigenvalues, we have to do the following. Let $\lambda = \alpha + i\,\beta$ be a complex eigenvalue with a set of linearly independent eigenvectors $\boldsymbol{x}_1 + i\,\boldsymbol{y}_1, \boldsymbol{x}_2 + i\,\boldsymbol{y}_2, \ldots$. Then we have to replace the $e^{\lambda t}(\boldsymbol{x}_i + i\,\boldsymbol{y}_i)$ and $e^{\bar{\lambda}}(\boldsymbol{x}_i - i\,\boldsymbol{y}_i)$ terms in the general solution formula by

$$e^{\alpha t}\left[(\cos\beta t)\,\boldsymbol{x}_i - (\sin\beta t)\,\boldsymbol{y}_i\right] \text{ and } e^{\alpha t}\left[(\sin\beta t)\,\boldsymbol{x}_i + (\cos\beta t)\,\boldsymbol{y}_i\right] \tag{46}$$

2. Note that here we have used implicitly the fact that eigenvectors corresponding to different eigenvalues are linearly independent. To prove this one needs to know that that the Vandermonde determinant

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \neq 0 \tag{43}$$

when all λ_i's are distinct. This fact can be proved in two ways. One way uses mathematical induction to prove that its determinant is in fact $\Pi(\lambda_i - \lambda_j)$; The other considers the following

$$\det\begin{pmatrix} 1 & \cdots & 1 & 1 \\ \lambda_1 & \cdots & \lambda_{n-1} & \lambda \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda^{n-1} \end{pmatrix} \tag{44}$$

which is clearly a polynomial of λ of degree at most $n-1$. Such a polynomial has at most $n-1$ roots. But clearly $\lambda_1, \ldots, \lambda_{n-1}$ are roots. Therefore the determinant is nonzero for any $\lambda \neq \lambda_1, \ldots, \lambda_{n-1}$.