Lecture 32 Vectors, Matrices, Determinants, Eigenvalues/Eigenvectors (Cont.)

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Matrices (cont.).

- Transpose; Conjugate; Adjoint

  From A we can form its

  - Transpose:

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} = (\, a_{ji}\,) = \begin{pmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix} = (\, \boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_m\,). \tag{1}$$

  - Conjugate:

$$\bar{A} = \begin{pmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \cdots & \bar{a}_{mn} \end{pmatrix} = (\bar{a}_{ij}) = (\, \bar{\boldsymbol{a}}_1 \ \cdots \ \bar{\boldsymbol{a}}_n\,) = \begin{pmatrix} \bar{\boldsymbol{b}}_1^T \\ \vdots \\ \bar{\boldsymbol{b}}_m^T \end{pmatrix}. \tag{2}$$

  - Adjoint:

$$A^* = (\bar{A}\,)^T = \left(\overline{A^T}\,\right) = (\bar{a}_{ji}) = \begin{pmatrix} \bar{\boldsymbol{a}}_1^T \\ \vdots \\ \bar{\boldsymbol{a}}_n^T \end{pmatrix} = \left(\, \bar{\boldsymbol{b}}_1 \ \cdots \ \bar{\boldsymbol{b}}_m\,\right). \tag{3}$$

- Identity

  The matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\delta_{ij}) \tag{4}$$

  is called the "identity matrix". Here $\delta_{ij} = 1$ when $i = j$ and 0 otherwise. We have[1]

$$I\,A = A; \qquad A\,I = A \tag{5}$$

  when the multiplications make sense.

- Inverse

  Inverse. For most $n \times n$ (square) matrices there exists a unique matrix such that their product is I.

  - This particular matrix is denoted A^{-1}. That is

$$A^{-1}A = A\,A^{-1} = I. \tag{6}$$

    Note that, for both products to make sense, A^{-1} has to be also $n \times n$.

  - Those matrices that have inverses are called "invertible" or "non-singular". The rest are called "singular".
  - A is singular $\Longleftrightarrow$ there is a column vector $\boldsymbol{x}$ (or a row vector $\boldsymbol{y}^T$) such that

$$A\,\boldsymbol{x} = 0\,(\text{or } \boldsymbol{y}^T A = 0) \tag{7}$$

    $\Longleftrightarrow$the columns of A are linearly dependent $\Longleftrightarrow$det $A = 0$ $\Longleftrightarrow$ the rows of A are linearly dependent.

- Matrix functions
  - Matrix functions.
    - Once each entry is a function of t, the matrix A becomes a "matrix function", denoted $A(t)$.

1. When A is an $m \times n$ matrix, the I in the first equality is the $m \times m$ identity matrix, while the I in the second equality is the $n \times n$ matrix.

- Differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \dot{A}(t) = A'(t) = (\dot{a}_{ij}(t)). \tag{8}$$

- System of DEs in vector-matrix form:

$$\dot{\boldsymbol{x}}\,(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{g}(t). \tag{9}$$

Here $\boldsymbol{x}, \boldsymbol{g}$ are n column vector, A is $n \times n$.

- When $\boldsymbol{g} = 0$, that is the homogeneous case, if we put n solutions $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ into a matrix:

$$X(t) = (\ \boldsymbol{x}_1(t) \ \ \cdots \ \ \boldsymbol{x}_n(t)\ ) \tag{10}$$

then one can check

$$\dot{X}\,(t) = A(t)\,X(t). \tag{11}$$

Eigenvalues/Eigenvectors.

- Eigenvalues:

  Let A be an $n \times n$ matrix. A number λ is said to be (one of) its eigenvalue if $A - \lambda I$ is singular.

- Eigenvectors: Those $\boldsymbol{x}$ such that

$$(A - \lambda\,I)\,\boldsymbol{x} = 0 \tag{12}$$

are called the (corresponding) eigenvectors.

Note that, if $\boldsymbol{x}$ is an eigenvector, so are $a\,\boldsymbol{x}$ for any number a. More generally, if $\boldsymbol{x}_1$, ..., $\boldsymbol{x}_k$ are eigenvectors corresponding to the same eigenvalue λ, so are $a_1\,\boldsymbol{x}_1 + \cdots + a_k\,\boldsymbol{x}_k$ for any numbers $a_1, \ldots,$ a_k.

- Recall formulas for determinants (for 2×2 and 3×3 matrices)

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}; \tag{13}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{32}\,a_{21} - a_{13}\,a_{22}\,a_{31} - a_{23}\,a_{32}\,a_{11} - a_{12}\,a_{21}\,a_{33}. \tag{14}$$

To remember the 2nd formula, write:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \tag{15}$$

put a + sign before all three "upper-left to down-right" products (such as $a_{11}\,a_{22}\,a_{33}$) and a − sign before all three "upper-right to down-left" products (such as $a_{11}\,a_{23}\,a_{32}$).

- Computation of eigenvalues/eigenvectors:

  1. Compute all the roots to $\det(A - \lambda\,I) = 0$. They are the eigenvalues.
  2. For each root λ_i, solve

$$(A - \lambda\,I)\,\boldsymbol{x} = 0. \tag{16}$$

Example. Find all eigenvalues and eigenvectors for $\left(\begin{smallmatrix} 5 & -1 \\ 3 & 1 \end{smallmatrix}\right)$.

Solution.

We compute

$$\det\left[\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)\,(1-\lambda) - (-1)\,3 = \lambda^2 - 6\,\lambda + 8. \tag{17}$$

Next solve

$$\lambda^2 - 6\,\lambda + 8 = 0 \Longrightarrow \lambda_1 = 2, \lambda_2 = 4. \tag{18}$$

Now find eigenvectors corresponding to $\lambda_1 = 2$:

$$A - \lambda_1 I = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}. \tag{19}$$

Now solve

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 3 \end{pmatrix}. \tag{20}$$

Similarly for $\lambda_2 = 4$, we compute

$$A - \lambda_2 I = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}. \tag{21}$$

Solving

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{22}$$

Example. Find all eigenvalues and eigenvectors for $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$.

Solution. We compute

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)^3 + 0\cdot(-2)\cdot 3 + 2\cdot 2\cdot 0 - 0\cdot(1-\lambda)\cdot 3 - 2\cdot 0\cdot(1-\lambda) - (-2)\cdot 2\cdot(1-\lambda) \\ &= (1-\lambda)\,[(1-\lambda)^2 + 4]. \end{aligned} \tag{23}$$

Solving

$$(1-\lambda)\,[(1-\lambda)^2 + 4] = 0 \tag{24}$$

we get three eigenvalues

$$\lambda_1 = 1, \lambda_2 = 1 + 2i, \lambda_3 = 1 - 2i. \tag{25}$$

- Eigenvectors corresponding to 1: Solve

  $$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{26}$$

  We use Gaussian elimination.

  $$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix} &\Longrightarrow \begin{pmatrix} 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\Longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(Simplify the 1st row)} \\ &\Longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(First row } \times(-3) \text{ add to 2nd)} \end{aligned} \tag{27}$$

  Thus we have

  $$x_1 - x_3 = 0 \tag{28}$$

  $$2x_2 + 3x_3 = 0 \tag{29}$$

  This gives

  $$x_1 = x_3, \qquad x_2 = -\frac{3}{2}x_3 \tag{30}$$

so any eigenvector can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -\frac{3}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}. \tag{31}$$

Here x_3 is free. Recalling the structure of the set of eigenvectors, we see that the eigenvectors corresponding to 1 are

$$a \begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix} \tag{32}$$

with an arbitrary constant a.

- Eigenvectors corresponding to $1 + 2\,i$: We need to solve

$$\begin{pmatrix} -2\,i & 0 & 0 \\ 2 & -2\,i & -2 \\ 3 & 2 & -2\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{33}$$

Gaussian elimination:

$$\begin{aligned} \begin{pmatrix} -2\,i & 0 & 0 & 0 \\ 2 & -2\,i & -2 & 0 \\ 3 & 2 & -2\,i & 0 \end{pmatrix} &\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -2\,i & -2 & 0 \\ 3 & 2 & -2\,i & 0 \end{pmatrix} \\ &\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2\,i & -2 & 0 \\ 0 & 2 & -2\,i & 0 \end{pmatrix} \\ &\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 2 & -2\,i & 0 \end{pmatrix} \\ &\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

So x_1, x_2, x_3 satisfy

$$x_1 = 0 \tag{35}$$

$$x_2 - i\,x_3 = 0 \tag{36}$$

so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -i\,x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}. \tag{37}$$

The eigenvectors corresponding to $1 + 2\,i$ are

$$a \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}. \tag{38}$$

- Eigenvectors corresponding to $1 - 2\,i$. Similar calculation gives[2]

$$a \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}. \tag{39}$$

2. In fact, one can show that if A is a real matrix, λ is an eigenvalue of A with corresponding eigenvector $\boldsymbol{x}$. Then $\bar{\lambda}$ is also an eigenvalue of A with corresponding eigenvector $\bar{\boldsymbol{x}}$.