Lecture 30 System of Ordinary Differential Equations

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Mathematical Modeling.

- A single differential equation models the time evolution of one quantity of interest.
- When there are two or more quantities of interest, we need two or more equations to model them.
- In general, these quantities will interact with one another, so the equations will be "coupled" – we cannot get information of any one quantity by looking at its equation alone, and have to treat the several equations as a whole, in other words, a system.
- For example of modeling, see the textbook, or my notes for last year's 334.

Theoretical Issues.

- **Reduction to First Order**
  - Any system of ordinary differential equations can be written as a bigger, but first order, system.
  - Example:

$$\ddot{x} = y^2 + (\dot{x})^3 + x \quad (1)$$
$$\dot{y} = x^3 \quad (2)$$

    becomes, after introducing $z = \dot{x}$,

$$\dot{z} = y^2 + z^3 + x \quad (3)$$
$$\dot{x} = z \quad (4)$$
$$\dot{y} = x^3 \quad (5)$$

- **Existence and Uniqueness**
  - Only need to consider the first order system:

$$\dot{x_1} = F_1(t, x_1, \ldots, x_n) \quad (6)$$
$$\dot{x_2} = F_2(t, x_1, \ldots, x_n) \quad (7)$$
$$\vdots \quad \vdots \quad \vdots$$
$$\dot{x}_n = F_n(t, x_1, \ldots, x_n) \quad (8)$$

    with initial conditions:

$$x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0. \quad (9)$$

  - Let R be a region in the t-x_1-x_2-$\cdots$-x_n space (which is $n+1$ dimensional) containing the point $(t_0, x_1^0, \ldots, x_n^0)$. If all the partial derivatives $\frac{\partial F_i}{\partial x_j}$ $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ remain bounded in R, then there **is** a **unique** solution $(x_1(t), \ldots, x_n(t))$ satisfying both the equations and the initial condition.
  - For example, for the system

$$\dot{z} = y^2 + z^3 + x \quad (10)$$
$$\dot{x} = z \quad (11)$$
$$\dot{y} = x^3 \quad (12)$$

    Take R to be any bounded domain in the four-dimensional space $\mathbb{R}^4$ (t, x, y, z), we compute all 9 partial derivatives

$$\frac{\partial(y^2 + z^3 + x)}{\partial z} = 3\,z^2, \quad \frac{\partial(y^2 + z^3 + x)}{\partial y} = 2\,y, \quad \frac{\partial(y^2 + z^3 + x)}{\partial x} = 1 \quad (13)$$

$$\frac{\partial z}{\partial z} = 1, \quad \frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial(x^3)}{\partial z} = 0, \quad \frac{\partial(x^3)}{\partial x} = 3\,x^2, \quad \frac{\partial(x^3)}{\partial y} = 0 \quad (14)$$

and see that all of them are bounded on R. Therefore this system has a unique solution given any initial point inside R.

First Order Linear Constant-coefficient System.

- Such system looks like

$$\dot{x}_1 = a_{11}\,x_1 + \cdots + a_{1n}\,x_n + g_1(t) \tag{15}$$
$$\vdots \quad \vdots \quad \vdots$$
$$\dot{x}_n = a_{n1}\,x_1 + \cdots + a_{nn}\,x_n + g_n(t). \tag{16}$$

  For example

$$\dot{x} = 3\,x + 2\,y + 5\,z + t^2 \tag{17}$$
$$\dot{y} = 2\,x + y + e^t \tag{18}$$
$$\dot{z} = 5\,x + 2\,y + 3\,z + t. \tag{19}$$

- Significance.
  - We have seen that any system can be reduced to first order.
  - The significance of linear constant-coefficient systems are two-fold:
    - For any system there are one or more linear systems which can describe the solution of the original nonlinear system around most important locations – the so-called "equilibrium points".
    - We can solve them completely.
- Basic Theory of first order linear system:

$$\dot{x}_1 = p_{11}(t)\,x_1 + \cdots + p_{1n}(t)\,x_n + g_1(t) \tag{20}$$
$$\vdots \quad \vdots \quad \vdots$$
$$\dot{x}_n = p_{n1}(t)\,x_1 + \cdots + p_{nn}(t)\,x_n + g_n(t). \tag{21}$$

  - General solution:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = c_1\,\boldsymbol{x}^{(1)}(t) + \cdots + c_n\,\boldsymbol{x}^{(n)}(t) + \boldsymbol{x}_p(t) \tag{22}$$

    with

$$\boldsymbol{x}^{(1)}(t) = \begin{pmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{pmatrix}, \ldots, \boldsymbol{x}^{(n)}(t) = \begin{pmatrix} x_1^{(n)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{pmatrix}, \quad \boldsymbol{x}_p(t) = \begin{pmatrix} x_{p1}(t) \\ \vdots \\ x_{pn}(t) \end{pmatrix}. \tag{23}$$

    Or equivalently:

$$x_1(t) = c_1\,x_1^{(1)}(t) + \cdots + c_n\,x_1^{(n)}(t) + x_{p1}(t), \tag{24}$$
$$\vdots$$
$$x_n(t) = c_1\,x_n^{(1)}(t) + \cdots + c_n\,x_n^{(n)}(t) + x_{pn}(t). \tag{25}$$

    In matrix form:

$$\boldsymbol{x}(t) = X(t)\,\boldsymbol{c} + \boldsymbol{x}_p \tag{26}$$

    where

$$X(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{27}$$

    and $\boldsymbol{x}, \boldsymbol{x}_p$ as defined previously.
  - $\boldsymbol{x}_p$ is a "particular solution", that is, any one solution of the system under consideration.

- $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ form a "fundamental set" for the corresponding homogeneous system:

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n \qquad (28)\\ \vdots \; &\vdots \; \vdots \\ \dot{x}_n &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n. \qquad (29) \end{aligned}$$

  That is, they are solutions, and they are linearly independent.

- Wronkian. The Wronskian of n solutions to the homogeneous system

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n \qquad (30)\\ \vdots \; &\vdots \; \vdots \\ \dot{x}_n &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n. \qquad (31) \end{aligned}$$

  is defined as

$$W\left[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}\right] = \det X(t) \qquad (32)$$

  with

$$\boldsymbol{x}^{(1)}(t) = \begin{pmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{pmatrix}, \ldots, \boldsymbol{x}^{(n)}(t) = \begin{pmatrix} x_1^{(n)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{pmatrix}, \quad X(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}. \qquad (33)$$

  - W satisfies

$$\frac{\mathrm{d}W}{\mathrm{d}t} = (p_{11}(t) + \cdots + p_{nn}(t))\, W. \qquad (34)$$

  - As a consequence Wronskian is either never 0 or is 0 for all t (as long as all the coefficients $p_{ij}(t)$ remain bounded)
  - $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ form a fundamental set if and only if their Wronskian is nonzero at the initial time t_0 (or any other single time).