LECTURE 29 CONVOLUTION

11/21/2011

Review.

- Solution procedure of Laplace transform method:
  1. Transform the equation: left hand side and the right hand side;
  2. Obtain $Y(s)$;
  3. Take inverse transform $y(t) = \mathcal{L}^{-1}\{Y\}$.
- What are involved in transforming the right hand side:
  1. Table of Laplace transform of basic functions;
  2. $\mathcal{L}\{f(t)\, u(t-a)\} = e^{-as}\, \mathcal{L}\{f(t+a)\}$ when the right hand side is piecewise defined;
  3. $\mathcal{L}\{f(t)\, \delta(t-a)\} = e^{-as}\, f(a)$ when the right hand side involves the impulse function.
- What are involved in the inverse transform:
  1. Table of Laplace transforms;
  2. $\mathcal{L}^{-1}\{e^{-as}\, F(s)\} = u(t-a)\, f(t-a)$.
- A remark about the smoothness of solution:
  1. When the right hand side is smooth, the solution is smooth;
  2. When the right hand side is piecewise defined, the solution is at least C^1 (meaning: y and y' are continuous functions);
  3. When the right hand side involves δ, the solution is continuous – y' is not continuous anymore.

Convolution.

- A "binary" operation: Two inputs and one output. Other examples of binary operations include $+, -, \times, /$. Each of them takes two functions as inputs and give out one single function as the output.
- Convolution is **much harder** to compute than the four basic operations. In particular, **don't confuse convolution with multiplication!**
- Given two functions $f(t)$ and $g(t)$, their convolution is defined as

$$h(t) := \{f * g\}(t) = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau. \tag{1}$$

- Basic properties of convolution:
  - $f * g = g * f$;
  - $f * (g_1 + g_2) = f * g_1 + f * g_2$;
  - $(f * g) * h = f * (g * h)$;
  - $f * 0 = 0 * f = 0$.

  Proofs of the above are left as homework.

  Note: Don't confuse convolution with multiplication! In particular, $f * 1 \neq f$!

  Example 1. Compute $e^t * 1$ and $\cos t * 1$.
  Solution. By definition we have

$$e^t * 1 = \int_0^t e^{\tau}\, 1\, \mathrm{d}\tau = e^t - 1; \tag{2}$$

$$(\cos t) * 1 = \int_0^t \cos(\tau)\, \mathrm{d}\tau = \sin t. \tag{3}$$

- Why bother with convolution:

Theorem 2. *If* $\mathcal{L}\{f\}=F, \mathcal{L}\{g\}=G$, *then* $\mathcal{L}^{-1}\{F(s)\,G(s)\}=f*g$.

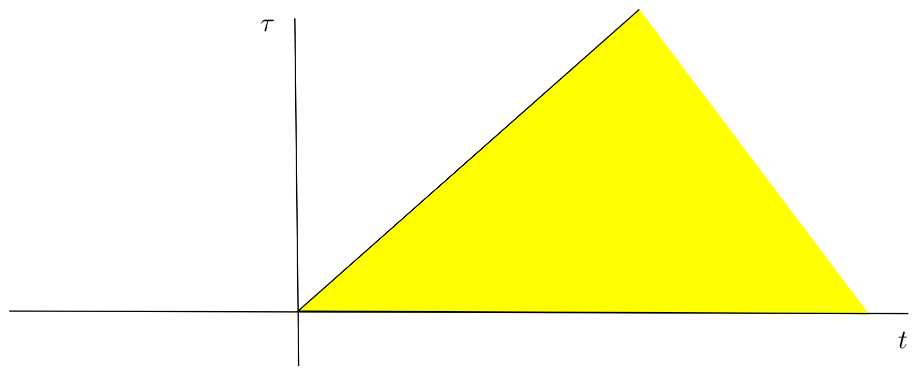
Proof. All we need to show is

$$\int_0^\infty e^{-st}\,(f*g)(t)\,\mathrm{d}t = F(s)\,G(s). \tag{4}$$

We have

$$\begin{aligned}\int_0^\infty e^{-st}\,(f*g)(t)\,\mathrm{d}t &= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau\right]\mathrm{d}t\\ &= \int_0^\infty \int_0^t [e^{-st} f(t-\tau)\,g(\tau)]\,\mathrm{d}\tau\,\mathrm{d}t.\end{aligned} \tag{5}$$

We now switch the order of the integration. To do this we have to first understand what the region of integration look like. We have t running from 0 to ∞, and τ from 0 to t. This is the triangular region $0<\tau<t<\infty$ in the $t-\tau$ plane:



To switch the order of integration, we first check what values can τ take? The answer is any value from 0 to ∞. Once τ is given, t can only run from τ to ∞. So we have

$$\int_0^\infty \int_0^t [e^{-st} f(t-\tau)\,g(\tau)]\,\mathrm{d}\tau\,\mathrm{d}t = \int_0^\infty \int_\tau^\infty [e^{-st} f(t-\tau)\,g(\tau)]\,\mathrm{d}t\,\mathrm{d}\tau. \tag{6}$$

Remark. Another way of doing this is to first observe: $0<t<\infty$ together with $0<\tau<t$ is the same as $0<\tau<t<\infty$. Now ignore t we have $0<\tau<\infty$, once τ is given, the range for t is $\tau<t<\infty$.

Further calculate:

$$\begin{aligned}\int_0^\infty \int_\tau^\infty [e^{-st} f(t-\tau)\,g(\tau)]\,\mathrm{d}t\,\mathrm{d}\tau &= \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-st} f(t-\tau)\,\mathrm{d}t\right]\mathrm{d}\tau\\ &= \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-s\tau} e^{-s(t-\tau)} f(t-\tau)\,\mathrm{d}t\right]\mathrm{d}\tau\\ &= \int_0^\infty e^{-s\tau} g(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)\,\mathrm{d}t\right]\mathrm{d}\tau.\end{aligned} \tag{7}$$

Setting $t'=t-\tau$ in the t-integral, we have

$$\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)\,\mathrm{d}t = \int_0^\infty e^{-st'} f(t')\,\mathrm{d}t' = F(s). \tag{8}$$

Therefore we reach

$$\int_0^\infty e^{-s\tau} g(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)\,\mathrm{d}t\right]\mathrm{d}\tau = \int_0^\infty e^{-s\tau} g(\tau)\,F(s)\,\mathrm{d}\tau = F(s)\,G(s) \tag{9}$$

and the proof ends. □

- Applications of convolution (From least to most important):
  - Find Laplace transforms for integrals:

    Example 3. Find the Laplace transform of $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, \mathrm{d}\tau$.
    Solution. First recognize:

    $$\int_0^t (t-\tau)^2 \cos 2\tau \, \mathrm{d}\tau = t^2 * (\cos 2t). \tag{10}$$

    Thus we have

    $$\begin{aligned} F(s) &= \mathcal{L}\{t^2 * (\cos 2t)\} \\ &= \mathcal{L}\{t^2\}\,\mathcal{L}\{\cos 2t\} \\ &= \frac{2}{s^3}\frac{s}{s^2+4} = \frac{2}{s^2(s^2+4)}. \end{aligned} \tag{11}$$

  - Alternative way of computing inverse Laplace transform:

    Example 4. Find the inverse Laplace transform of $\frac{2}{s^2(s^2+4)}$.
    Solution. Write

    $$\frac{2}{s^2(s^2+4)} = \frac{1}{s^2}\frac{2}{s^2+4}. \tag{12}$$

    Thus we have

    $$\mathcal{L}^{-1}\left\{\frac{2}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = t * (\sin 2t). \tag{13}$$

    Note: It's **not OK** to stop here!

    Compute

    $$\begin{aligned} t * (\sin 2t) &= \int_0^t (t-\tau)\sin(2\tau)\,\mathrm{d}\tau \\ &= -\frac{1}{2}\int_0^t (t-\tau)\,\mathrm{d}\cos(2\tau) \\ &= -\frac{1}{2}\left[(t-\tau)\cos(2\tau)|_{\tau=0}^{\tau=t} - \int_0^t \cos(2\tau)\,\mathrm{d}(t-\tau)\right] \\ &= -\frac{1}{2}\left[-t + \int_0^t \cos(2\tau)\,\mathrm{d}\tau\right] \\ &= \frac{t}{2} - \frac{1}{4}\sin(2t). \end{aligned} \tag{14}$$

    Check

    $$\mathcal{L}\left\{\frac{t}{2} - \frac{1}{4}\sin(2t)\right\} = \frac{1}{2s^2} - \frac{1}{4}\frac{2}{s^2+4} = \frac{2}{s^2(s^2+4)}. \tag{15}$$

  - Obtain solution formula for general right hand sides.

    Example 5. Express the solution of the following initial value problem in terms of a convolution integral:

    $$y'' + 4y' + 4y = g(t); \qquad y(0) = 0, y'(0) = 0. \tag{16}$$

    Solution.

    First transform the equation:

    $$\mathcal{L}\{y''\} = s^2 Y - s\,y(0) - y'(0) = s^2 Y; \tag{17}$$

    $$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y \tag{18}$$

    Denoting $\mathcal{L}\{g\} = G(s)$, we have the transformed equation as

    $$(s^2 + 4s + 4)\,Y = G(s). \tag{19}$$

So

$$Y = \frac{G(s)}{s^2 + 4\,s + 4}. \tag{20}$$

Now take inverse transform:

$$y = \mathcal{L}^{-1}\left\{\frac{G(s)}{(s+2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} * \mathcal{L}^{-1}\{G\} = (e^{-2t} \quad t) * g = \int_0^t e^{-2(t-\tau)} \quad (t - \tau)\, g(\tau)\, \mathrm{d}\tau. \tag{21}$$

Remark 6. Once we obtain this formula, we can get some information of the solution without really compute them. For example, if we are further told that g is bounded, that is there is a constant M such that $|g| \leqslant M$, then we can immediately compute

$$|y| \leqslant M \left| \int_0^t e^{-2(t-\tau)}\,(t-\tau)\,\mathrm{d}\tau \right| = M \left[1 - \frac{2\,t+1}{4}\,e^{-2t}\right]. \tag{22}$$

In particular we can conclude $|y|$ share the same bound M as g without requiring detailed information of g.