Lecture 28 Impulse Functions

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Impulse functions.

- Impulse functions are functions that are zero everywhere except at one single points, and furthermore when integrated over $\mathbb{R}$ give a nonzero value.
- **Unit impulse function** (guess this is engineering jargon. In physics it's known as the Dirac δ-function, and in mathematics just δ).

  $\delta(t)=0$ for all $t\neq 0$, and $\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t=1$.
- Translations:

$$\delta(t-a)=0 \text{ for all } t\neq a, \qquad \int_{-\infty}^{\infty}\delta(t-a)=1. \tag{1}$$

- Properties.
  - It's easy to see that

$$\int_{t_1}^{t_2}\delta(t-a)\,\mathrm{d}x=\begin{cases}0 & a<t_1 \text{ or } a>t_2\\ 1 & t_1<a<t_2\end{cases} \tag{2}$$

    Remark. What if $t_1=a$ or $t_2=a$? In these cases we have to clarify whether our $\delta(t-a)$ means $\delta(t-(a+))$ – a unit impulse just to the right of a, $\delta(t-(a-))$ – a unit impulse just to the left of a, or $\delta(t-a)$ that is a unit impulse sit right at a. The reasonable values of $\int_a^{t_2}\delta(t-a)\,\mathrm{d}x$ are then

$$\int_a^{t_2}\delta(t-(a+))=1; \qquad \int_a^{t_2}\delta(t-(a-))=0; \qquad \int_a^{t_2}\delta(t-a)=\frac{1}{2}. \tag{3}$$

    Similar argument works if $t_2=a$.

  - More generally we have, for any continuous function f,

$$\int_{t_1}^{t_2}\delta(t-a)f(t)\,\mathrm{d}t=\begin{cases}0 & a<t_1 \text{ or } a>t_2\\ f(a) & t_1<a<t_2\end{cases} \tag{4}$$

    To see why, consider the following argument:

    - The first case is obvious;
    - To prove the second case, first realize what continuity of f (at a) means: Given any $\varepsilon>0$, there is $\delta>0$ such that when $|t-a|<\delta$, $|f(t)-f(a)|<\varepsilon$.
    - Now consider a given $\varepsilon>0$. Let δ be as in above. If $|t_1-a|$ or $|t_2-a|$ is even smaller than δ, change δ to that smaller value.
    - Now compute

$$\begin{aligned}\int_{t_1}^{t_2}\delta(t-a)f(t)\,\mathrm{d}t &= \int_{a+\delta}^{t_2}\delta(t-a)\,f(t)\,\mathrm{d}t\\ &\quad+\int_{a-\delta}^{a+\delta}\delta(t-a)\,f(t)\,\mathrm{d}t\\ &\quad+\int_{t_1}^{a-\delta}\delta(t-a)\,f(t)\,\mathrm{d}t\\ &= 0+\int_{a-\delta}^{a+\delta}\delta(t-a)\,f(t)\,\mathrm{d}t+0\\ &= \int_{a-\delta}^{a+\delta}\delta(t-a)\,f(t)\,\mathrm{d}t\end{aligned}$$

$$= \int_{a-\delta}^{a+\delta} \delta(t-a)\, f(a)\, \mathrm{d}t + \int_{a-\delta}^{a+\delta} \delta(t-a)\, (f(t)-f(a))\, \mathrm{d}t$$
$$= f(a) \int_{a-\delta}^{a+\delta} \delta(t-a)\, \mathrm{d}t + \int_{a-\delta}^{a+\delta} \delta(t-a)\, (f(t)-f(a))\, \mathrm{d}t$$
$$= f(a) + \int_{a-\delta}^{a+\delta} \delta(t-a)\, (f(t)-f(a))\, \mathrm{d}t. \tag{5}$$

– Finally notice:

$$\left| \int_{a-\delta}^{a+\delta} \delta(t-a)\, (f(t)-f(a))\, \mathrm{d}t \right| \leqslant \int_{a-\delta}^{a+\delta} \delta(t-a)\, |f(t)-f(a)| < \varepsilon \int_{a-\delta}^{a+\delta} \delta(t-a) = \varepsilon \tag{6}$$

– Summarizing what we have shown:

$$\left| \int_{t_1}^{t_2} \delta(t-a) f(t)\, \mathrm{d}t - f(a) \right| < \varepsilon. \tag{7}$$

As ε can be any positive number, it must be true that

$$\left| \int_{t_1}^{t_2} \delta(t-a) f(t)\, \mathrm{d}t - f(a) \right| = 0. \tag{8}$$

Laplace transform of impulse functions.

In our class we only consider $a \geqslant 0$. Since all we care about is what happens when $t \geqslant 0$.

- Clearly e^{-st} is continuous for every s. So

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\, \delta(t-a)\, \mathrm{d}t = e^{-as}. \tag{9}$$

- More generally, if $f(t)$ is continuous, then so is $e^{-st} f(t)$ and we have

$$\mathcal{L}\{f(t)\, \delta(t-a)\} = \int_0^\infty e^{-st} f(t)\, \delta\,(t-a)\, \mathrm{d}t = e^{-as} f(a). \tag{10}$$

Example 1. Compute

$$\mathcal{L}\left\{e^{-t^2} \sin\left(\sqrt{3} \ln(\cos t)\right) \delta(t-1)\right\}. \tag{11}$$

Solution. Identify

$$f(t) = e^{-t^2} \sin\left(\sqrt{3} \ln(\cos t)\right), \qquad a = 1. \tag{12}$$

So

$$\mathcal{L}\left\{e^{-t^2} \sin\left(\sqrt{3} \ln(\cos t)\right) \delta(t-1)\right\} = e^{-as} f(a) = e^{-s} e^{-1} \sin\left(\sqrt{3} \ln(\cos 1)\right). \tag{13}$$

Equations with impulse right hand side.

Example 2. Solve

$$y'' + 4\, y = \delta(t-\pi) - \delta(t - 2\,\pi), \qquad y(0) = 0, \quad y'(0) = 0. \tag{14}$$

Solution.

First take Laplace transform of the equation: Recall

$$\mathcal{L}\{y''\} = s^2\, Y - s\, y(0) - y'(0). \tag{15}$$

So the transformed equation is

$$(s^2 + 4)\, Y = e^{-\pi s} - e^{-2\pi s}. \tag{16}$$

Next find Y:

$$Y = \frac{e^{-\pi s}}{s^2 + 4} - \frac{e^{-2\pi s}}{s^2 + 4}. \tag{17}$$

Finally take the inverse transform: Recall

$$\mathcal{L}^{-1}\{e^{-a s} F(s)\} = u(t-a)\, f(t-a) \tag{18}$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

- $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\}$: Identify $a = \pi, F(s) = \frac{1}{s^2+4}$. So $f(t) = \frac{1}{2}\sin(2\,t)$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = u(t-\pi)\, f(t-\pi) = u(t-\pi)\,\frac{1}{2}\sin(2\,(t-\pi)) = \frac{1}{2}\,u(t-\pi)\sin(2\,t). \tag{19}$$

- $\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\} = \frac{1}{2}\,u(t-2\,\pi)\sin(2\,t)$.

So finally

$$y = \frac{1}{2}\sin(2\,t)\,[u(t-\pi) - u(t-2\,\pi)]. \tag{20}$$

To understand the solution better, we can further write it into the piecewise form:

$$y = \begin{cases} 0 & t < \pi \\ \frac{1}{2}\sin(2\,t) & \pi < t < 2\,\pi \\ 0 & t > 2\,\pi \end{cases}. \tag{21}$$

Example 3. Solve

$$2\,y'' + y' + 4\,y = \delta\left(t - \frac{\pi}{6}\right)\sin(t), \qquad y(0) = y'(0) = 0. \tag{22}$$

Solution. First recall

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0); \tag{23}$$

$$\mathcal{L}\{y'\} = s\,Y - y(0) \tag{24}$$

$$\mathcal{L}\left\{\delta\left(t - \frac{\pi}{6}\right)\sin(t)\right\} = e^{-\frac{\pi}{6}s}\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\,e^{-\frac{\pi}{6}s}. \tag{25}$$

Now we can write down the transformed equation

$$(2\,s^2 + s + 4)\,Y = \frac{1}{2}\,e^{-\frac{\pi}{6}s}. \tag{26}$$

This gives

$$Y(s) = \frac{1}{2}\,\frac{e^{-\frac{\pi}{6}s}}{2\,s^2 + s + 4}. \tag{27}$$

To compute inverse transform recall

$$\mathcal{L}^{-1}\{e^{-a s} F(s)\} = u(t-a)\, f(t-a). \tag{28}$$

Identify:

$$F(s) = \frac{1}{2}\,\frac{1}{2\,s^2 + s + 4}; \qquad a = \frac{\pi}{6}. \tag{29}$$

We need to compute

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{1}{2\,s^2 + s + 4}\right\} \\ &= \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + s/2 + 2}\right\} \\ &= \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{31}{16}}\right\} \\ &= \frac{1}{4}\,e^{-t/4}\,\frac{4}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{4}\,t\right) \\ &= \frac{1}{\sqrt{31}}\,e^{-t/4}\sin\left(\frac{\sqrt{31}}{4}\,t\right). \end{aligned}$$

Now we can write

$$y = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{e^{-\frac{\pi}{6}s}}{2s^2+s+4}\right\} = u\left(t-\frac{\pi}{6}\right)\frac{1}{\sqrt{31}}e^{-\left(t-\frac{\pi}{6}\right)/4}\sin\left(\frac{\sqrt{31}}{4}\left(t-\frac{\pi}{6}\right)\right). \qquad (30)$$