Lecture 27 Solving DEs involving Jumps

11/16/2011

Review.

- One can use "unit step function"
$$u(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{1}$$
to represent functions with jumps.
- (New: added Nov. 18) Translations of $u(t)$:
$$u(t-a) := \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \tag{2}$$
In many books $u(t-a)$ is often denoted in the more compact form $u_a(t)$.
- The representation of a function
$$g(t) = \begin{cases} g_1(t) & 0 < t < t_1 \\ \vdots & \\ g_k(t) & t_{k-1} < t < t_k \end{cases} \tag{3}$$
is
$$g(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t-t_1) + [g_3(t) - g_2(t)]\, u(t-t_2) + \cdots + [g_k(t) - g_{k-1}(t)]\, u(t-t_{k-1}). \tag{4}$$
- Calculating $\mathcal{L}\{g(t)\, u(t-a)\}$:
  1. Obtain $f(t) = g(t+a)$;
  2. Compute $F(s) = \mathcal{L}\{f\}$.
  3. Multiply it by e^{-as} to get $\mathcal{L}\{g(t)\, u(t-a)\} = e^{-as}\, F(s)$.
- Calculating $\mathcal{L}^{-1}\{e^{-as}\, F(s)\}$:
  1. Identify a;
  2. Compute $f(t) = \mathcal{L}^{-1}\{F\}$.
  3. Replace every t by $t-a$ in $f(t)$ to get $f(t-a)$.
  4. Multiply it by $u(t-a)$ to finally obtain
$$\mathcal{L}^{-1}\{e^{-as}\, F(s)\} = f(t-a)\, u(t-a). \tag{5}$$

Examples.

Now we solve equations.

Example 1. Solve
$$y'' + y = u(t-3); \qquad y(0) = 0, \quad y'(0) = 1. \tag{6}$$

Solution. We follow the same old three steps.

1. Transform the equation. Compute
$$\mathcal{L}\{y''\} = s^2\, Y - s\, y(0) - y'(0) = s^2\, Y - 1. \tag{7}$$
Thus the equation is transformed into
$$(s^2 + 1)\, Y = \frac{e^{-3s}}{s} + 1. \tag{8}$$
2. Solve the transformed equation.
$$Y = \frac{e^{-3s}}{s\,(s^2+1)} + \frac{1}{s^2+1}. \tag{9}$$

3. Transform back. Compute

$$y=\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s\,(s^2+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = f(t-3)\,u(t-3)+\sin t. \tag{10}$$

Here

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}. \tag{11}$$

Now we compute the inverse transform of $\frac{1}{s\,(s^2+1)}$ via method of partial fraction:

$$\frac{1}{s\,(s^2+1)}=\frac{A}{s}+\frac{B\,s+C}{s^2+1}=\frac{A\,(s^2+1)+(B\,s+C)\,s}{s\,(s^2+1)}. \tag{12}$$

So

$$A\,(s^2+1)+(B\,s+C)\,s=1 \tag{13}$$

which leads to

$$A+B = 0 \tag{14}$$
$$C = 0 \tag{15}$$
$$A = 1 \tag{16}$$

Therefore

$$A=1, B=-1, C=0 \tag{17}$$

and

$$\frac{1}{s\,(s^2+1)}=\frac{1}{s}-\frac{s}{s^2+1}. \tag{18}$$

So

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=1-\cos t. \tag{19}$$

This gives

$$f(t-3)=1-\cos\,(t-3) \tag{20}$$

and finally

$$y=[1-\cos\,(t-3)]\,u(t-3)+\sin t. \tag{21}$$

Example 2. Solve

$$y''+3\,y'+2\,y=u_2(t); \qquad y(0)=0, y'(0)=1. \tag{22}$$

Solution.

First transform the equation:

$$\mathcal{L}\{y''\} = s^2\,Y-s\,y(0)-y'(0)=s^2\,Y-1; \tag{23}$$
$$\mathcal{L}\{y'\} = s\,Y-y(0)=s\,Y \tag{24}$$
$$\mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s}. \tag{25}$$

So the transformed equation is

$$(s^2+3\,s+2)\,Y-1=\frac{e^{-2s}}{s} \tag{26}$$

which gives

$$Y=\frac{1}{s^2+3\,s+2}+\frac{e^{-2s}}{s\,(s^2+3\,s+2)}. \tag{27}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3\,s+2}\right\}$.

  Factorize:

$$s^2+3\,s+2=(s+2)\,(s+1) \tag{28}$$

so

$$\frac{1}{s^2+3\,s+2}=\frac{A}{s+2}+\frac{B}{s+1} \tag{29}$$

which leads to

$$1=A\,(s+1)+B\,(s+2)\Longrightarrow A=-1, B=1. \tag{30}$$

So

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3\,s+2}\right\}=-\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}=-e^{-2t}+e^{-t}. \tag{31}$$

- $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s\,(s^2+3\,s+2)}\right\}$.

  Due to the presence of e^{-2s} we know that $u(t-2)$ is involved. In the formula

$$\mathcal{L}^{-1}\{e^{-a\,s}\,F(s)\}=f(t-a)\,u(t-a). \tag{32}$$

we have $a=2, F(s)=\frac{1}{s\,(s^2+3\,s+2)}$.

First we find $f(t)=\mathcal{L}^{-1}\{F\}$. The partial fraction formula should be:

$$\frac{1}{s\,(s^2+3\,s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}=\frac{A\,(s+1)\,(s+2)+B\,s\,(s+2)+C\,s\,(s+1)}{s\,(s+1)\,(s+2)}. \tag{33}$$

This gives

$$1=(A+B+C)\,s^2+(3\,A+2\,B+C)\,s+2\,A \tag{34}$$

therefore

$$A+B+C = 0 \tag{35}$$

$$3\,A+2\,B+C = 0 \tag{36}$$

$$2\,A = 1 \tag{37}$$

whose solutions are

$$A=\frac{1}{2},\ B=-1,\ C=\frac{1}{2}. \tag{38}$$

Therefore

$$\frac{1}{s\,(s^2+3\,s+2)}=\frac{1/2}{s}-\frac{1}{s+1}+\frac{1/2}{s+2} \tag{39}$$

and

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+3\,s+2)}\right\}=\frac{1}{2}-e^{-t}+\frac{1}{2}\,e^{-2t}. \tag{40}$$

This gives

$$f(t-2)=\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}\,e^{-2(t-2)}. \tag{41}$$

So we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s\,(s^2+3\,s+2)}\right\}=\left[\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}\,e^{-2(t-2)}\right]u(t-2). \tag{42}$$

Putting things together, we have

$$y=\left[\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}\,e^{-2(t-2)}\right]u(t-2)-e^{-2t}+e^{-t} \tag{43}$$

Example 3. Solve

$$y''+5\,y'+6\,y=g(t), \qquad y(0)=0, \quad y'(0)=2, \tag{44}$$

where

$$g(t)=\begin{cases} 0 & 0\leqslant t<1 \\ t & 1<t<5 \\ 1 & 5<t \end{cases}. \tag{45}$$

Solution.

1. Transform the equation.

   a. Transform the LHS. Compute

$$\mathcal{L}\{y''\} = s^2 Y - s\,y(0) - y'(0) = s^2 Y - 2; \tag{46}$$

$$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y. \tag{47}$$

   So

$$\mathcal{L}\{y'' + 5\,y' + 6\,y\} = (s^2 + 5\,s + 6)\,Y - 2. \tag{48}$$

   b. Transform the RHS. We first need to represent g using unit step functions. There are two discontinuities at $t = 1, 5$. So

$$g(t) = t\,u(t-1) + (1-t)\,u(t-5). \tag{49}$$

   Recalling

$$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-a s}\,\mathcal{L}\{g(t+a)\}(s) \tag{50}$$

   We compute

$$\begin{aligned}\mathcal{L}\{t\,u(t-1) + (1-t)\,u(t-5)\} &= e^{-s}\,\mathcal{L}\{t+1\} + e^{-5s}\,\mathcal{L}\{-t-4\} \\ &= e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{4}{s}\right).\end{aligned} \tag{51}$$

   Thus the transformed equation reads

$$(s^2 + 5\,s + 6)\,Y = e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + 2. \tag{52}$$

2. Solve the transformed equation. We have

$$Y(s) = e^{-s}\,\frac{s+1}{s^2\,(s^2 + 5\,s + 6)} - e^{-5s}\,\frac{4\,s+1}{s^2\,(s^2 + 5\,s + 6)} + \frac{2}{s^2 + 5\,s + 6}. \tag{53}$$

3. Transform back. We need to use the formula

$$\mathcal{L}^{-1}\{e^{-a s}\,F(s)\} = f(t-a)\,u(t-a). \tag{54}$$

   Thus we have to compute the inverse transforms of $\frac{s+1}{s^2\,(s^2+5\,s+6)}$ and $\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}$. Due to linearity of the inverse transform, we only need to compute

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2 + 5\,s + 6)}\right\} \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2 + 5\,s + 6)}\right\} \tag{55}$$

   as

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2\,(s^2 + 5\,s + 6)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2 + 5\,s + 6)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2 + 5\,s + 6)}\right\} \tag{56}$$

   and

$$\mathcal{L}^{-1}\left\{\frac{4\,s+1}{s^2\,(s^2 + 5\,s + 6)}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2 + 5\,s + 6)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2 + 5\,s + 6)}\right\} \tag{57}$$

   - $\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2+5\,s+6)}\right\}$.

     Factorize

$$s^2\,(s^2 + 5\,s + 6) = s^2\,(s+2)\,(s+3). \tag{58}$$

     So

$$\frac{s}{s^2\,(s^2 + 5\,s + 6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}. \tag{59}$$

     This gives

$$s = A\,s\,(s+2)\,(s+3) + B\,(s+2)\,(s+3) + C\,s^2\,(s+3) + D\,s^2\,(s+2). \tag{60}$$

The right hand side expands to

$$(A+C+D)\,s^3+(5\,A+B+3\,C+2\,D)\,s^2+(6\,A+5\,B)\,s+6\,B. \tag{61}$$

Thus

$$\begin{aligned} A+C+D &= 0 & (62)\\ 5\,A+B+3\,C+2\,D &= 0 & (63)\\ 6\,A+5\,B &= 1 & (64)\\ 6\,B &= 0 & (65)\end{aligned}$$

The solutions are

$$A=\frac{1}{6},\ B=0,\ C=-\frac{1}{2},\ D=\frac{1}{3}. \tag{66}$$

So

$$\frac{s}{s^2\,(s^2+5\,s+6)}=\frac{1/6}{s}-\frac{1/2}{s+2}+\frac{1/3}{s+3} \tag{67}$$

which gives

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2+5\,s+6)}\right\}=\frac{1}{6}-\frac{1}{2}\,e^{-2t}+\frac{1}{3}\,e^{-3t}. \tag{68}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2+5\,s+6)}\right\}$. We write

$$\frac{1}{s^2\,(s^2+5\,s+6)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}+\frac{D}{s+3} \tag{69}$$

which leads to

$$1=(A+C+D)\,s^3+(5\,A+B+3\,C+2\,D)\,s^2+(6\,A+5\,B)\,s+6\,B \tag{70}$$

and consequently

$$\begin{aligned} A+C+D &= 0 & (71)\\ 5\,A+B+3\,C+2\,D &= 0 & (72)\\ 6\,A+5\,B &= 0 & (73)\\ 6\,B &= 1 & (74)\end{aligned}$$

So

$$A=-\frac{5}{36},\ B=\frac{1}{6},\ C=\frac{1}{4},\ D=-\frac{1}{9}. \tag{75}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{1}{6}\,t+\frac{1}{4}\,e^{-2t}-\frac{1}{9}\,e^{-3t}-\frac{5}{36}. \tag{76}$$

Thus we have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{1}{36}+\frac{t}{6}-\frac{1}{4}\,e^{-2t}+\frac{2}{9}\,e^{-3t} \tag{77}$$

which leads to

$$\mathcal{L}^{-1}\left\{e^{-s}\,\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\}=\left(\frac{1}{36}+\frac{t-1}{6}-\frac{1}{4}\,e^{-2(t-1)}+\frac{2}{9}\,e^{-3(t-1)}\right)u(t-1); \tag{78}$$

For the other term we have

$$\mathcal{L}^{-1}\left\{\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{19}{36}+\frac{t}{6}-\frac{7}{4}\,e^{-2t}+\frac{11}{9}\,e^{-3t} \tag{79}$$

which leads to

$$\mathcal{L}^{-1}\left\{e^{-5s}\,\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}\right\}=\left(\frac{19}{36}+\frac{t-5}{6}-\frac{7}{4}\,e^{-2(t-5)}+\frac{11}{9}\,e^{-3(t-5)}\right)u(t-5). \tag{80}$$

Finally we quickly compute

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+5\,s+6}\right\}=2\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}=2\,e^{-2t}-2\,e^{-3t}. \tag{81}$$

Summarizing, we reach

$$\begin{aligned} y(t) &= \left(\frac{1}{36} + \frac{t-1}{6} - \frac{1}{4}\,e^{-2(t-1)} + \frac{2}{9}\,e^{-3(t-1)}\right) u(t-1) \\ &\quad - \left(\frac{19}{36} + \frac{t-5}{6} - \frac{7}{4}\,e^{-2(t-5)} + \frac{11}{9}\,e^{-3(t-5)}\right) u(t-5) + 2\,e^{-2t} - 2\,e^{-3t}. \end{aligned} \tag{82}$$