Last time we mentioned the necessity of considering functions with jumps, such as:

\[ g(t) = \begin{cases} 
1 & 0 < t < 3 \\
0 & t \geq 3 
\end{cases} \]  

(1)

Now if we only want to do the Laplace transform of this function, then definition is enough:

\[
\mathcal{L}\{g\}(s) = \int_0^\infty e^{-st} g(t) \, dt = \int_0^3 e^{-st} \, dt + \int_3^\infty te^{-st} \, dt = \frac{1}{s} + \left( \frac{2}{s^2} - \frac{1}{s^3} \right) e^{-3s}.
\]  

(2)

However, imagine the following situation. After transforming an equation, we get

\[ Y = \frac{1}{s} + \left( \frac{2}{s^2} - \frac{1}{s^3} \right) e^{-3s}. \]

(3)

How can we possibly figure out

\[ y(t) = \begin{cases} 
1 & 0 < t < 3 \\
0 & t \geq 3 
\end{cases} \]  

(4)

Therefore we need a more systematic way of dealing with Laplace and inverse Laplace transforms involving step functions.

Fortunately such a way exists. The key is the “unit step function”

\[ u(t) := \begin{cases} 
0 & t < 0 \\
1 & t \geq 0 
\end{cases} \]  

(5)

**Unit step function and representation of functions with jumps.**

- The unit step function

\[ u(t) := \begin{cases} 
0 & t < 0 \\
1 & t \geq 0 
\end{cases} \]  

(6)

represents a jump of unit size at \( t = 0 \).

- Notice the following: If we translate \( u(t) \) by \( a \), that is replace \( t \) by \( t - a \), where \( a \) is any number, then the function

\[ u(t-a) = \begin{cases} 
0 & t < a \\
1 & t \geq a 
\end{cases} \]  

(7)

represents a jump of unit size at \( t = a \). Note that \( u(t-a) \) is sometimes denoted by \( u_a(t) \).

- One step further, we realize that

\[ Mu(t-a) = \begin{cases} 
0 & t < a \\
M & t \geq a 
\end{cases} \]  

(8)

represents a jump of size \( M \) at \( t = a \). Therefore a “jump” of any size at anywhere can be thus represented.

- With the help of \( u(t) \) and its translations, we are able to “decompose” any functions with jumps into a sum of terms like

\[ u(t-a)g(t) \]  

(9)

where \( g(t) \) is a nice function.

- More specifically, the representation of a function

\[ g(t) = \begin{cases} 
g_1(t) & 0 < t < t_1 \\
\vdots 
g_k(t) & t_{k-1} < t < t_k 
\end{cases} \]  

(10)
Example 1. Express the given function using unit step functions.

\[
g(t) = \begin{cases} 
  0 & 0 < t < 1 \\
  2 & 1 < t < 2 \\
  1 & 2 < t < 3 \\
  3 & 3 < t
\end{cases}
\]  

Solution. We have \( g_1(t) = 0 \), \( g_2(t) = 2 \), \( g_3(t) = 1 \), \( g_4(t) = 3 \). Thus

\[
g(t) = 2u(t - 1) - u(t - 2) + 2u(t - 3).
\]

Example 2. Express

\[
g(t) = \begin{cases} 
  0 & 0 < t < 2 \\
  t + 1 & 2 < t
\end{cases}
\]  

using unit jump function.

Solution. We have

\[
g(t) = (t + 1)u(t - 2).
\]

- Of course we can also recover \( g \). For example, if we are given

\[
g(t) = 2u(t - 1) - u(t - 2) + 2u(t - 3).
\]

and would like to get a “piecewise” formula, we do the following.

1. Identify the “jump” points: 1, 2, 3. This means the formula for \( g \) would look like

\[
g(t) = \begin{cases} 
  g_1(t) & 0 < t < 1 \\
  g_2(t) & 1 < t < 2 \\
  g_3(t) & 2 < t < 3 \\
  g_4(t) & 3 < t
\end{cases}
\]

2. Now recall the formula

\[
g(t) = g_1(t) + [g_2(t) - g_1(t)]u(t - t_1) + [g_3(t) - g_2(t)]u(t - t_2) + \cdots + [g_k(t) - g_{k-1}(t)]u(t - t_{k-1}).
\]

Comparing with

\[
g(t) = 2u(t - 1) - u(t - 2) + 2u(t - 3) = 0 + 2u(t - 1) - u(t - 2) + 2u(t - 3)
\]

we have

\[
\begin{align*}
g_1(t) &= 0 \\
g_2(t) - g_1(t) &= 2 \implies g_2(t) = 2 \\
g_3(t) - g_2(t) &= -1 \implies g_3(t) = 1 \\
g_4(t) - g_3(t) &= 2 \implies g_4(t) = 3.
\end{align*}
\]

Thus we recover

\[
g(t) = \begin{cases} 
  0 & 0 < t < 1 \\
  2 & 1 < t < 2 \\
  1 & 2 < t < 3 \\
  3 & 3 < t
\end{cases}
\]

- If we are asked to a function with formula involving \( u \):

1. Obtain the piecewise formula using the above procedure;
2. Draw the plot.

Laplace Transform of functions with Jumps.

- Laplace transform of \( u(t - a) g(t) \).

\[
\mathcal{L} \{ g(t) u(t - a) \} = \int_0^\infty e^{-st} g(t) u(t - a) \, dt = \int_a^\infty e^{-st} g(t) \, dt = e^{-as} \int_a^\infty e^{-(s - a) t} g(t) \, dt = e^{-as} \int_a^\infty e^{-sv} g(v + a) \, dv = e^{-as} \mathcal{L} \{ g(t + a) \}(s). \tag{24}
\]

In particular, we have

\[
\mathcal{L} \{ u(t - a) \} = \frac{e^{-as}}{s}. \tag{25}
\]

- Therefore to evaluate \( \mathcal{L} \{ g(t) u(t - a) \} \) we have to do the following:
  1. Obtain \( f(t) = g(t + a) \).
  2. Compute \( F(s) = \mathcal{L} \{ f \} \).
  3. Multiply it by \( e^{-as} \) to get \( \mathcal{L} \{ g(t) u(t - a) \} = e^{-as} F(s) \).

Example 3. Compute the Laplace transform of

\[
g(t) = \begin{cases} 
0 & 0 < t < 1 \\
2 & 1 < t < 2 \\
1 & 2 < t < 3 \\
3 & 3 < t 
\end{cases}. \tag{26}
\]

Solution. We have already found out that

\[
g(t) = 2u(t - 1) - u(t - 2) + 2u(t - 3). \tag{27}
\]

Thus

\[
\mathcal{L} \{ g \}(s) = 2 \mathcal{L} \{ u(t - 1) \} - \mathcal{L} \{ u(t - 2) \} + 2 \mathcal{L} \{ u(t - 3) \} = \frac{2e^{-s} - e^{-2s} + 2e^{3s}}{s}. \tag{28}
\]

Example 4. Compute Laplace transform of

\[
g(t) = \begin{cases} 
0 & 0 < t < 2 \\
t + 1 & 2 < t 
\end{cases}. \tag{29}
\]

Solution. We have already solved

\[
g(t) = (t + 1) u(t - 2). \tag{30}
\]

Let \( \tilde{g}(t) = t + 1 \). We have

\[
\mathcal{L} \{ \tilde{g}(t) u(t - 2) \} = e^{-2s} \mathcal{L} \{ \tilde{g}(t + 2) \} = e^{-2s} \mathcal{L} \{ t + 3 \} = e^{-2s} \left[ \frac{1}{s^2} + \frac{3}{s} \right]. \tag{31}
\]

Inverse transform.

- Inverse transform. We observe that the universal character of Laplace transforms of functions with jumps is the appearance of \( e^{-as} \). So all we need is a formula for \( \mathcal{L}^{-1} \{ e^{-as} F(s) \} \). Since

\[
f(t - a) = g(t), \tag{32}
\]

means

\[
g(t + a) = f(t) \tag{33}
\]
the formula
\[ \mathcal{L}\{g(t) u(t-a)\} = e^{-as} \mathcal{L}\{g(t+a)\}(s) \] (34)
can be written as
\[ \mathcal{L}\{f(t) u(t-a)\} = e^{-as} F(s). \] (35)
So we reach
\[ \mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a). \] (36)

- Therefore to compute \( \mathcal{L}^{-1}\{e^{-as} F(s)\} \) we need to do the following:
  1. Identify \( a \);
  2. Compute \( f(t) = \mathcal{L}^{-1}\{F\} \).
  3. Replace every \( t \) by \( t-a \) in \( f(t) \) to get \( f(t-a) \).
  4. Multiply it by \( u(t-a) \) to finally obtain
\[ \mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a). \] (37)

**Example 5.** Determine the inverse Laplace transform of
\[ \frac{e^{-2s}}{s-1}. \] (38)

**Solution.** We have
\[ \mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a). \] (39)
Comparing with the problem, we have \( a = 2 \), and \( F(s) = \frac{1}{s-1} \). Inverting \( F(s) \) we have
\[ f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t. \] (40)
Thus
\[ f(t-2) = e^{t-2}. \] (41)

So finally
\[ \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} = e^{t-2} u(t-2). \] (42)

**Example 6.** Compute the inverse Laplace transform of
\[ \frac{s e^{-3s}}{s^2 + 4s + 5}. \] (43)

**Solution.** Comparing with
\[ \mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a). \] (44)
we have \( a = 3 \), \( F(s) = \frac{s}{s^2 + 4s + 5} \). We compute
\[
\begin{align*}
f(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\} \\
    &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\right\} \\
    &= e^{-2t} [\cos t - 2 \sin t].
\end{align*}
\] (45)

Thus
\[
\begin{align*}
\mathcal{L}^{-1}\left\{\frac{s e^{-3s}}{s^2 + 4s + 5}\right\} &= f(t-3) u(t-3) \\
    &= e^{-2(t-3)} [\cos (t - 3) - 2 \sin (t - 3)] u(t-3).
\end{align*}
\] (46)