Lecture 26 Step Functions

11/14/2011

Last time we mentioned the necessity of considering functions with jumps, such as:

$$g(t) = \begin{cases} 1 & 0 < t < 3 \\ t & t > 3 \end{cases}. \tag{1}$$

Now if we only want to do the Laplace transform of this function, then definition is enough:

$$\mathcal{L}\{g\}(s) = \int_0^\infty e^{-st} g(t)\,\mathrm{d}t = \int_0^3 e^{-st}\,\mathrm{d}t + \int_3^\infty t\,e^{-st}\,\mathrm{d}t = \frac{1}{s} + \left(\frac{2}{s} - \frac{1}{s^2}\right) e^{-3s}. \tag{2}$$

However, imagine the following situation. After transforming an equation, we get

$$Y = \frac{1}{s} + \left(\frac{2}{s} - \frac{1}{s^2}\right) e^{-3s}. \tag{3}$$

How can we possibly figure out

$$y(t) = \begin{cases} 1 & 0 < t < 3 \\ t & t > 3 \end{cases} ? \tag{4}$$

Therefore we need a more systematic way of dealing with Laplace and inverse Laplace transforms involving step functions.

Fortunately such a way exists. The key is the "unit step function"

$$u(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}. \tag{5}$$

Unit step function and representation of functions with jumps.

- The unit step function

$$u(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}. \tag{6}$$

  represents a jump of unit size at $t = 0$.

- Notice the following: If we translate $u(t)$ by a, that is replace t by $t - a$, where a is any number, then the function

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \tag{7}$$

  represents a jump of unit size at $t = a$. Note that $u(t-a)$ is sometimes denoted by $u_a(t)$.

- One step further, we realize that

$$M\,u(t-a) = \begin{cases} 0 & t < a \\ M & t > a \end{cases} \tag{8}$$

  represents a jump of size M at $t = a$. Therefore a "jump" of any size at anywhere can be thus represented.

- With the help of $u(t)$ and its translations, we are able to "decompose" any functions with jumps into a sum of terms like

$$u(t-a)\,g(t) \tag{9}$$

  where $g(t)$ is a nice function.

- More specifically, the representation of a function

$$g(t) = \begin{cases} g_1(t) & 0 < t < t_1 \\ \vdots & \\ g_k(t) & t_{k-1} < t < t_k \end{cases} \tag{10}$$

is

$$g(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t - t_1) + [g_3(t) - g_2(t)]\, u(t - t_2) + \cdots + [g_k(t) - g_{k-1}(t)]\, u(t - t_{k-1}). \tag{11}$$

Example 1. Express the given function using unit step functions.

$$g(t) = \begin{cases} 0 & 0 < t < 1 \\ 2 & 1 < t < 2 \\ 1 & 2 < t < 3 \\ 3 & 3 < t \end{cases}. \tag{12}$$

Solution. We have $g_1(t) = 0,\ g_2(t) = 2,\ g_3(t) = 1,\ g_4(t) = 3$. Thus

$$g(t) = 2\, u(t-1) - u(t-2) + 2\, u(t-3).$$

Example 2. Express

$$g(t) = \begin{cases} 0 & 0 < t < 2 \\ t+1 & 2 < t \end{cases} \tag{13}$$

using unit jump function.

Solution. We have

$$g(t) = (t+1)\, u(t-2). \tag{14}$$

- Of course we can also recover g. For example, if we are given

$$g(t) = 2\, u(t-1) - u(t-2) + 2\, u(t-3). \tag{15}$$

and would like to get a "piecewise" formula, we do the following.

1. Identify the "jump" points: $1, 2, 3$. This means the formula for g would look like

$$g(t) = \begin{cases} g_1(t) & 0 < t < 1 \\ g_2(t) & 1 < t < 2 \\ g_3(t) & 2 < t < 3 \\ g_4(t) & 3 < t \end{cases}. \tag{16}$$

2. Now recall the formula

$$g(t) = g_1(t) + [g_2(t) - g_1(t)]\, u(t - t_1) + [g_3(t) - g_2(t)]\, u(t - t_2) + \cdots + [g_k(t) - g_{k-1}(t)]\, u(t - t_{k-1}). \tag{17}$$

Comparing with

$$g(t) = 2\, u(t-1) - u(t-2) + 2\, u(t-3) = 0 + 2\, u(t-1) - u(t-2) + 2\, u(t-3) \tag{18}$$

we have

$$\begin{aligned} g_1(t) &= 0 && (19)\\ g_2(t) - g_1(t) &= 2 \Longrightarrow g_2(t) = 2 && (20)\\ g_3(t) - g_2(t) &= -1 \Longrightarrow g_3(t) = 1 && (21)\\ g_4(t) - g_3(t) &= 2 \Longrightarrow g_4(t) = 3. && (22) \end{aligned}$$

Thus we recover

$$g(t) = \begin{cases} 0 & 0 < t < 1 \\ 2 & 1 < t < 2 \\ 1 & 2 < t < 3 \\ 3 & 3 < t \end{cases}. \tag{23}$$

- If we are asked to a function with formula involving u:

1. Obtain the piecewise formula using the above procedure;

2. Draw the plot.

Laplace Transform of functions with Jumps.

- Laplace transform of $u(t-a)\,g(t)$.

$$\begin{aligned}\mathcal{L}\{g(t)\,u(t-a)\} &= \int_0^\infty e^{-st}\,g(t)\,u(t-a)\,\mathrm{d}t \\ &= \int_a^\infty e^{-st}\,g(t)\,\mathrm{d}t \\ &= e^{-as}\int_a^\infty e^{-s(t-a)}\,g(t)\,\mathrm{d}t \\ &= e^{-as}\int_a^\infty e^{-sv}\,g(v+a)\,\mathrm{d}v \\ &= e^{-as}\,\mathcal{L}\{g(t+a)\}(s). \end{aligned} \tag{24}$$

In particular, we have

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}. \tag{25}$$

- Therefore to evaluate $\mathcal{L}\{g(t)\,u(t-a)\}$ we have to do the following:
  1. Obtain $f(t) = g(t+a)$;
  2. Compute $F(s) = \mathcal{L}\{f\}$.
  3. Multiply it by e^{-as} to get $\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,F(s)$.

Example 3. Compute the Laplace transform of

$$g(t) = \begin{cases} 0 & 0<t<1 \\ 2 & 1<t<2 \\ 1 & 2<t<3 \\ 3 & 3<t \end{cases}. \tag{26}$$

Solution. We have already found out that

$$g(t) = 2\,u(t-1) - u(t-2) + 2\,u(t-3). \tag{27}$$

Thus

$$\mathcal{L}\{g\}(s) = 2\,\mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} + 2\,\mathcal{L}\{u(t-3)\} = \frac{2\,e^{-s} - e^{-2s} + 2\,e^{3s}}{s}. \tag{28}$$

Example 4. Compute Laplace transform of

$$g(t) = \begin{cases} 0 & 0<t<2 \\ t+1 & 2<t \end{cases} \tag{29}$$

Solution. We have already solved

$$g(t) = (t+1)\,u(t-2). \tag{30}$$

Let $\tilde{g}(t) = t+1$. We have

$$\mathcal{L}\{\tilde{g}(t)\,u(t-2)\} = e^{-2s}\,\mathcal{L}\{\tilde{g}(t+2)\} = e^{-2s}\,\mathcal{L}\{t+3\} = e^{-2s}\left[\frac{1}{s^2} + \frac{3}{s}\right]. \tag{31}$$

Inverse transform.

- Inverse transform. We observe that the universal character of Laplace transforms of functions with jumps is the appearance of e^{-as}. So all we need is a formula for $\mathcal{L}^{-1}\{e^{-as}\,F(s)\}$. Since

$$f(t-a) = g(t), \tag{32}$$

means

$$g(t+a) = f(t) \tag{33}$$

the formula

$$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-a s}\,\mathcal{L}\{g(t+a)\}(s) \tag{34}$$

can be written as

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-a s}\,F(s). \tag{35}$$

So we reach

$$\mathcal{L}^{-1}\{e^{-a s}\,F(s)\} = f(t-a)\,u(t-a). \tag{36}$$

- Therefore to compute $\mathcal{L}^{-1}\{e^{-a s}\,F(s)\}$ we need to do the following:
  1. Identify a;
  2. Compute $f(t) = \mathcal{L}^{-1}\{F\}$.
  3. Replace every t by $t-a$ in $f(t)$ to get $f(t-a)$.
  4. Multiply it by $u(t-a)$ to finally obtain

$$\mathcal{L}^{-1}\{e^{-a s}\,F(s)\} = f(t-a)\,u(t-a). \tag{37}$$

Example 5. Determine the inverse Laplace transform of

$$\frac{e^{-2s}}{s-1}. \tag{38}$$

Solution. We have

$$\mathcal{L}^{-1}\{e^{-a s}\,F(s)\} = f(t-a)\,u(t-a). \tag{39}$$

Comparing with the problem, we have $a=2$, and $F(s) = \frac{1}{s-1}$. Inverting $F(s)$ we have

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t. \tag{40}$$

Thus

$$f(t-2) = e^{t-2}. \tag{41}$$

So finally

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} = e^{t-2}\,u(t-2). \tag{42}$$

Example 6. Compute the inverse Laplace transform of

$$\frac{s\,e^{-3s}}{s^2+4\,s+5}. \tag{43}$$

Solution. Comparing with

$$\mathcal{L}^{-1}\{e^{-a s}\,F(s)\} = f(t-a)\,u(t-a). \tag{44}$$

we have $a=3$, $F(s) = \frac{s}{s^2+4\,s+5}$. We compute

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4\,s+5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\right\} \\ &= e^{-2t}\,[\cos t - 2\sin t]. \end{aligned} \tag{45}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s\,e^{-3s}}{s^2+4\,s+5}\right\} &= f(t-3)\,u(t-3) \\ &= e^{-2(t-3)}\,[\cos(t-3) - 2\sin(t-3)]\,u(t-3). \end{aligned} \tag{46}$$