Lecture 24 Solve Differential Equations (Cont.)

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Solving differential equations using Laplace Transform.

- 3 Steps:

  1. Transform the equation:

     ◦ Transform the left hand side. Tools:

       – $\mathcal{L}\{y^{(n)}\} = s^n Y - s^{n-1} y(0) - \cdots - y^{(n-1)}(0)$;

       – Linearity: $\mathcal{L}\{a_n\ y^{(n)} + a_{n-1}\ y^{(n-1)} + \cdots + a_0\ y\} = a_n\ \mathcal{L}\{y^{(n)}\} + a_{n-1}\mathcal{L}\{y^{(n-1)}\} + \cdots + a_0 Y$.

     ◦ Transform the right hand side. Tools:

       – Basic formulas:

$$\mathcal{L}\{1\} = \frac{1}{s}; \tag{1}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}; \tag{2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}; \tag{3}$$

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2} \tag{4}$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} \tag{5}$$

       – Properties: In the following $F(s) = \mathcal{L}\{f\}$.

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \tag{6}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s) \tag{7}$$

       – Linearity.

  2. Get $Y(s) = \mathcal{L}\{y\}(s)$.

     ◦ Usually trivial.

  3. Figure out y such that $y(t) = \mathcal{L}^{-1}\{Y\}$. Tools:

     ◦ Partial fraction. If $Y = \frac{P}{Q}$[1] where P, Q are polynomials, write Y as sum of terms of the form

$$\frac{A}{(s-a)^k}; \qquad \frac{B(s-a)+C}{(s-a)^2+b^2}; \qquad \frac{B(s-a)+C}{[(s-a)^2+b^2]^k}. \tag{8}$$

     ◦ Linearity: In the following $F(s) = \mathcal{L}\{f\}$.

$$\mathcal{L}^{-1}\{aF + bG\} = af + bg. \tag{9}$$

1. In the following lectures we will see how to deal with the more general situation $Y = e^{ct}\frac{P}{Q}$.

- Inverse transform formulas: In the following $F(s) = \mathcal{L}\{f\}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{10}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \tag{11}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos b\,t \tag{12}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin b\,t \tag{13}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t) \tag{14}$$

$$\mathcal{L}^{-1}\left\{(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)\right\} = t^n f(t) \tag{15}$$

For example, to compute

$$\mathcal{L}^{-1}\left\{\frac{B\,(s-a)+C}{(s-a)^2+b^2}\right\} \tag{16}$$

we first use linearity:

$$\mathcal{L}^{-1}\left\{\frac{B\,(s-a)+C}{(s-a)^2+b^2}\right\} = B\,\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} + \frac{C}{b}\,\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}. \tag{17}$$

we notice that

$$\frac{s-a}{(s-a)^2+b^2} = F(s-a) \tag{18}$$

for

$$F(s) = \frac{s}{s^2+b^2}. \tag{19}$$

Since we know $\mathcal{L}^{-1}\{F\} = \cos b\,t$, we use $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$ to get

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos b\,t. \tag{20}$$

Similarly we get

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin b\,t. \tag{21}$$

So finally

$$\mathcal{L}^{-1}\left\{\frac{B\,(s-a)+C}{(s-a)^2+b^2}\right\} = B\,e^{at}\cos b\,t + \frac{C}{b}\,e^{at}\sin b\,t. \tag{22}$$

Note. $\mathcal{L}^{-1}\left\{\frac{B\,(s-a)+C}{[(s-a)^2+b^2]^k}\right\}$ is quite complicated and is not required in this class.

Examples.

Example 1. Solve

$$y^{(4)} - y = 0; \qquad y(0) = 1,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 0. \tag{23}$$

Solution.

1. Transform the equation.

$$\begin{aligned}\mathcal{L}\{y^{(4)}\} &= s^4\,Y - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0) \\ &= s^4\,Y - s^3 - s;\end{aligned} \tag{24}$$

So the transformed equation is

$$[s^4\,Y - s^3 - s] - Y = 0 \tag{25}$$

which simplifies to

$$(s^4 - 1)\,Y = s^3 + s. \tag{26}$$

2. Find Y. We have

$$Y(s) = \frac{s^3+s}{s^4-1} = \frac{s\,(s^2+1)}{(s^2-1)\,(s^2+1)} = \frac{s}{s^2-1}. \tag{27}$$

3. Inverse transform of Y.

   First we do partial fraction:

$$\frac{s}{s^2-1} = \frac{s}{(s-1)\,(s+1)} \tag{28}$$

so the correct form of it partial fraction representation is

$$\frac{A}{s-1} + \frac{B}{s+1} = \frac{A\,(s+1) + B\,(s-1)}{(s-1)\,(s+1)}. \tag{29}$$

Equating

$$\frac{s}{(s-1)\,(s+1)} = \frac{A\,(s+1) + B\,(s-1)}{(s-1)\,(s+1)} \tag{30}$$

gives

$$s = A\,(s+1) + B\,(s-1) \Longrightarrow A + B = 1;\, A - B = 0. \tag{31}$$

So we have

$$A = B = \frac{1}{2}. \tag{32}$$

The partial fraction representation is

$$Y = \frac{1}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s+1}. \tag{33}$$

The inverse transform can then be computed as (using linearity and $\mathcal{L}^{-1}\{F(s-a)\} = e^{a\,t}\,f(t)$):

$$y = \mathcal{L}^{-1}\{Y\} = \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = \frac{e^t}{2} + \frac{e^{-t}}{2}. \tag{34}$$

Example 2. Solve

$$y'' - 2\,y' + 2\,y = e^{-t}, \qquad y(0) = 0, \quad y'(0) = 1. \tag{35}$$

Solution.

1. Transform the equation:

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y - 1; \tag{36}$$

$$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y. \tag{37}$$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s-(-1)} = \frac{1}{s+1}.$$

So the transformed equation is

$$[s^2\,Y - 1] - 2\,[s\,Y] + 2\,Y = \frac{1}{s+1}. \tag{38}$$

This simplifies to

$$(s^2 - 2\,s + 2)\,Y = \frac{1}{s+1} + 1. \tag{39}$$

2. Find Y.

$$Y = \frac{1}{(s^2 - 2\,s + 2)\,(s+1)} + \frac{1}{s^2 - 2\,s + 2}. \tag{40}$$

3. Inverse transform.

   By linearity we have

$$y = \mathcal{L}^{-1}\left\{\frac{1}{(s^2 - 2\,s + 2)\,(s+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2\,s + 2}\right\}. \tag{41}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{(s^2 - 2\,s + 2)\,(s+1)}\right\}$.

  We use partial fraction. First factorize the denominator:

$$(s^2 - 2\,s + 2)\,(s+1) = [(s-1)^2 + 1]\,(s+1) \tag{42}$$

cannot be further factorized. Therefore

$$\frac{1}{(s^2-2\,s+2)\,(s+1)}=\frac{A}{s+1}+\frac{B\,(s-1)+C}{s^2-2\,s+2}. \tag{43}$$

This leads to

$$1=A\,(s^2-2\,s+2)+[B\,(s-1)+C]\,(s+1) \tag{44}$$

which becomes

$$1=(A+B)\,s^2+(C-2\,A)\,s+2\,A+C-B. \tag{45}$$

So

$$A+B = 0 \tag{46}$$

$$C-2\,A = 0 \tag{47}$$

$$2\,A+C-B = 1. \tag{48}$$

Solving it gives

$$A=\frac{1}{5}; \qquad B=-\frac{1}{5}; \qquad C=\frac{2}{5}. \tag{49}$$

So

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2\,s+2)\,(s+1)}\right\}=\frac{e^{-t}}{5}-\frac{e^t\cos t}{5}+\frac{2\,e^t\sin t}{5}. \tag{50}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2\,s+2}\right\}$.

  We have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2\,s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}=e^t\sin t. \tag{51}$$

Putting things together we have

$$y=\frac{e^{-t}}{5}-\frac{e^t\cos t}{5}+\frac{7\,e^t\sin t}{5}. \tag{52}$$

Remark 3. A short comparison of Laplace transform method and undetermined coefficientsf[2].

- Applicability: Und. Coeff. is wider (so far).

  Both deal with left hand sides with constant coefficients, while right hand sides a sum of terms of the form $e^{at}\,t^n$, $e^{at}\,t^n\cos b\,t$, $e^{at}\,t^n\sin b\,t$. But undetermined coefficient can solve both initial value and boundary value problems, while Laplace transform becomes awkward when facing boundary value problems.

- Convenience: Laplace transform slightly easier.

  When applying undetermined coefficients one has to do very complicated calculation differentiating the guessed solution, and furthermore after getting the general solution one needs to solve an $n\times n$ linear system. On the other hand, no messy differentiation is needed when using Laplace transform. Although in Laplace transform method one also needs to solve an $n\times n$ system (when doing partial fraction), in many cases the calculation is simpler due to various tricks and possible cancellation (such as the cancellation in the first example).

Piecewise continuous and δ functions.

Another situation where the Laplace transform method is superior is when the right hand side involves functions that are not smooth or even nor continuous.

For example,

$$y''+5\,y'+6\,y=g(t), \qquad y(0)=0, \quad y'(0)=2, \tag{53}$$

where

$$g(t)=\begin{cases} 0 & 0\leqslant t<1 \\ t & 1<t<5 \\ 1 & 5<t \end{cases}. \tag{54}$$

2. Thanks to everyone for participation in the short discussion today.

A more exotic equation:

$$y'' + 2\,y' - 3\,y = \delta(t-1) - \delta(t-2). \qquad y(0) = 2, \quad y'(0) = -2. \tag{55}$$

Here $\delta(t-a)$ is the δ function at a: It is 0 for all $t \neq a$ while ∞ at $t = a$, and the ∞ is such that $\int_{t_1}^{t_2} \delta(t-a) = 1$ whenever $t_1 < a < t_2$.

Clearly, to be able to solve such systems, we need to be able to

1. Transform such functions;
2. Recognize the Laplace transform of such functions so that we can get the solution even if it's discontinuous.

perform Laplace transform on the above types of functions. We study them one by one.

Remark 4. The physical meaning of such right hand sides are as follows: Consider an object under forcing.

$$m\,\ddot{x} = F(t). \tag{56}$$

It may happen that

- The forcing suddenly changes. Say $F(t)$ is 0 for $t < 1$, but starts to increase rapidly at $t = 0$, and reaches 1 at $t = 1.0001$. A good way to mathematically model such forcing is to write

$$F(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & t > 1 \end{cases}. \tag{57}$$

  Such F is not continuous anymore, it has "jumps".

- The forcing is asserted in an impulsive way, say the object is hit by a hammer at $t = 1$. Thus the force is 0 for $t < 1$ and $t > 1.0001$, but is very large between $t = 1$ and $t = 1.0001$. Furthermore, in such cases it is reasonable to assume that the "impulse":

$$\int_1^{1.0001} F(t)\,\mathrm{d}t \tag{58}$$

  is a fixed number, say 1. A good way to model such forcing is to write

$$F(t) = \delta(t-1) \tag{59}$$

  where $\delta(t)$ is a "generalized function", satisfying

$$\delta(t) = 0 \text{ whenever } t \neq 0; \qquad \int_{-\varepsilon}^{\varepsilon} \delta(t)\,\mathrm{d}t = 1 \text{ for any } \varepsilon > 0. \tag{60}$$