Lecture 22 Laplace Transform

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One word about checking regular singular points.

- We should check analyticity of $(x-x_0)\,p$ and $(x-x_0)^2\,q$. For example,

$$y'' + \frac{1}{x\,(x-1)^2}\,y' + y = 0. \tag{1}$$

  Here $p = \frac{1}{x\,(x-1)^2}$, $q = 1$. Singular points are $x = 0$ and $x = 1$.

  - Check whether $x = 0$ is regular singular:

$$(x-0)\,p = \frac{1}{(x-1)^2}; \qquad (x-0)^2\,q = x^2. \tag{2}$$

    Both analytic at 0. So 0 is a regular singular point.

  - Check whether $x = 1$ is regular singular:

$$(x-1)\,p = \frac{1}{x\,(x-1)}, \qquad (x-1)^2\,q = (x-1)^2. \tag{3}$$

    We see that $(x-1)\,p$ is not analytic at $x = 1$. So 1 is an irregular singular point.

Definition of Laplace transform.

Definition 1. *Let $f(t)$ be a function on $[0,\infty)$. The Laplace transform of f is the function F defined by the integral*

$$\mathcal{L}\{f\}(s) := F(s) := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t. \tag{4}$$

Remark 2. Here $\mathcal{L}\{f\}(s)$ and $F(s)$ are two different notations of the same thing. The former is usually used when dealing with specific functions, while the latter is advantageous in a more abstract setting, in particular when unknown functions are involved. For example, if we take Laplace transform of $y'' + 3\,y' + 4\,y = f(t)$ where f denotes a generic function, writing the result as

$$(s^2 + 3\,s + 4)\,Y = F + s\,y(0) + y'(0) + 3\,y(0) \tag{5}$$

is much more convenient than using $\mathcal{L}\{y\}(s)$ instead of $Y(s)$.

On the other hand, the latter notation cannot deal with denoting the transform of a specific function, such as $\sin 3t$. While the first has no difficulty here.

Example 3. Compute the Laplace transform of the following functions.

$$f(t) = 1, e^{at}, t^n, \sin b\,t, \cos b\,t, e^{at}\,t^n, e^{at}\sin b\,t, e^{at}\cos b\,t. \tag{6}$$

Solution.

1. $f(t) = 1$. We compute

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}\,\mathrm{d}t. \tag{7}$$

   Clearly the integral is not finite for $s \leqslant 0$. For $s > 0$, We have

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = -\frac{1}{s}\,e^{-st}|_0^\infty = \frac{1}{s}. \tag{8}$$

2. $f(t) = e^{at}$. We compute

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{at}\,e^{-st}\,\mathrm{d}t = \int_0^\infty e^{(a-s)t}\,\mathrm{d}t = \frac{1}{s-a}. \tag{9}$$

The domain is $s > a$.

3. $f(t) = t^n$, $n = 1, 2, \ldots.$ Clearly we need to require $s > 0$, otherwise the integral is not finite.
   Compute

$$\begin{aligned} \mathcal{L}\{t^n\}(s) &= \int_0^\infty t^n e^{-st}\, \mathrm{d}t \\ &= -\frac{1}{s}\int_0^\infty t^n\, \mathrm{d}e^{-st} \\ &= -\frac{1}{s} t^n e^{-st}|_0^\infty + \frac{1}{s}\int e^{-st}\, \mathrm{d}t^n \\ &= \frac{n}{s}\int t^{n-1} e^{-st}\, \mathrm{d}t \\ &= \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s). \end{aligned} \tag{10}$$

Replacing n by $n-1$ we have

$$\mathcal{L}\{t^{n-1}\}(s) = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}(s). \tag{11}$$

Thus we have

$$\mathcal{L}\{t^n\}(s) = \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s) = \frac{n\,(n-1)}{s^2}\mathcal{L}\{t^{n-2}\}(s) = \cdots = \frac{n!}{s^n}\mathcal{L}\{t^0\}(s) = \frac{n!}{s^{n+1}}. \tag{12}$$

The domain is $s > 0$.

4. $f(t) = \sin b\,t$. Again we need to require $s > 0$ as otherwise the integral does not exist. We compute

$$\begin{aligned} \mathcal{L}\{\sin b\,t\}(s) &= \int_0^\infty \sin b\,t\, e^{-st}\, \mathrm{d}t \\ &= -\frac{1}{s}\int_0^\infty \sin b\,t\, \mathrm{d}e^{-st} \\ &= -\frac{1}{s}\sin b\,t\, e^{-st}|_0^\infty + \frac{1}{s}\int e^{-st}\, \mathrm{d}\sin b\,t \\ &= 0 + \frac{b}{s}\int e^{-st}\cos b\,t\, \mathrm{d}t \\ &= -\frac{b}{s^2}\int \cos b\,t\, \mathrm{d}e^{-st} \\ &= -\frac{b}{s^2}\left[\cos b\,t\, e^{-st}|_0^\infty - \int e^{-st}\, \mathrm{d}\cos b\,t\right] \\ &= -\frac{b}{s^2}\left[-1 + b\int e^{-st}\sin b\,t\right] \\ &= \frac{b}{s^2} - \frac{b^2}{s^2}\mathcal{L}\{\sin b\,t\}(s). \end{aligned} \tag{13}$$

This gives

$$\mathcal{L}\{\sin b\,t\}(s) = \frac{b}{s^2 + b^2}, \qquad s > 0. \tag{14}$$

5. $f(t) = \cos b\,t$. We can proceed similarly. But a quicker way is to notice that in the calculation of $\mathcal{L}\{\sin b\,t\}(s)$ we already obtain

$$\mathcal{L}\{\sin b\,t\}(s) = \frac{b}{s}\int e^{-st}\cos b\,t\, \mathrm{d}t = \frac{b}{s}\mathcal{L}\{\cos b\,t\}(s). \tag{15}$$

Thus

$$\mathcal{L}\{\cos b\,t\}(s) = \frac{s}{s^2 + b^2}, \qquad s > 0. \tag{16}$$

6. $f(t) = e^{at}\, t^n$, $n = 1, 2, \ldots$ We can compute using definition, but a quicker way is to notice that

$$\mathcal{L}\{e^{at}\, t^n\}(s) = \int_0^\infty e^{-(s-a)t}\, t^n\, \mathrm{d}t. \tag{17}$$

This is exactly the formula for $\mathcal{L}\{t^n\}$ with s replaced by $s-a$. Replacing every s by $s-a$ in the t^n case, we have

$$\mathcal{L}\{e^{at}\,t^n\}(s)=\mathcal{L}\{t^n\}(s-a)=\frac{n!}{(s-a)^{n+1}}. \tag{18}$$

Naturally, the domain changes from $s>0$ to $s-a>0$, or $s>a$.

7. $f(t)=e^{at}\sin b\,t$. Similarly, we conclude

$$\mathcal{L}\{e^{at}\sin b\,t\}(s)=\mathcal{L}\{\sin b\,t\}(s-a)=\frac{b}{(s-a)^2+b^2} \tag{19}$$

with domain $s>a$.

8. $f(t)=e^{at}\cos b\,t$. Similarly we obtain

$$\mathcal{L}\{e^{at}\cos b\,t\}=\frac{s}{(s-a)^2+b^2}. \tag{20}$$

Remark 4. In the above calculation we have done many integration by parts and have thrown away all the boundary terms are $t=\infty$. Clearly this is not OK for all values of s. The set of s where such operation is OK is called the "domain" of the transformed function. So for example, rigorously speaking, the Laplace transform of the function t is

$$\frac{1}{s^2}\text{ in the domain } s>0. \tag{21}$$

The following theorem gives us a way to determine the domain without calculating the integrals.

Theorem 5. *If $|f|\leqslant K\,e^{at}$ for all t, then $\mathcal{L}\{f\}(s)$ is defined for $s>a$. Or equivalently, the domain of $\mathcal{L}\{f\}$ contains the set $s>a$.*

In practice we have to figure out precisely the set of a such that the relation is true.

Example 6. What is the domain for $\mathcal{L}\{t^3\sin t\}$.

The key is to figure out all a's such that

$$|t^3\sin t|\leqslant K\,e^{at} \tag{22}$$

is true for some constant K. We know that any $a>0$ would do. On the other hand, notice that the left hand side is unbounded as $t\nearrow\infty$, while the right hand side remains bounded if $a\leqslant 0$, we conclude that any $a\leqslant 0$ does not work. Therefore the domain is the union of all $s>a$ for all $a>0$, which is $s>0$.

Properties of Laplace transform.

- Linearity. Let a,b be constants. Then

$$\mathcal{L}\{a\,f+b\,g\}=a\,\mathcal{L}\{f\}+b\,\mathcal{L}\{g\}. \tag{23}$$

- Transform of derivatives. We have

$$\mathcal{L}\left\{f^{(n)}\right\}(s)=s^n\,\mathcal{L}(f)(s)-s^{n-1}\,f(0)-s^{n-2}\,f'(0)-\cdots-f^{(n-1)}(0). \tag{24}$$

We will see this is the key of the power of Laplace transform method.

- Transform of products. There is a way to obtain $\mathcal{L}\{f\,g\}$ using $\mathcal{L}\{f\}$ and $\mathcal{L}\{g\}$ but it involves much calculation. However when one of f,g is e^{at} or t^n, we have the following:
  - $\mathcal{L}\{e^{at}\,f\}=F(s-a)$. Here $F(s)=\mathcal{L}\{f\}(s)$.

    For example, to compute $\mathcal{L}\{e^{at}\,t^n\}$, we identify

$$f(t)=t^n\Longrightarrow F(s)=\frac{n!}{s^{n+1}}. \tag{25}$$

So

$$\mathcal{L}\{e^{at}\,t^n\}=\frac{n!}{(s-a)^{n+1}}. \tag{26}$$

  - $\mathcal{L}\{t^n\,f\}=(-1)^n\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}F(s)$. Again $F(s)=\mathcal{L}\{f\}(s)$.

    For example, we can compute $\mathcal{L}\{t^n\}$ through identifying $f=1\Longrightarrow F(s)=\frac{1}{s}$. So

$$\mathcal{L}\{t^n\}=(-1)^n\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\frac{1}{s}\right)=(-1)^n(-1)^n\,n!\,s^{-(n+1)}=\frac{n!}{s^{n+1}}. \tag{27}$$