Lecture 19 Power Series Method (Cont.)

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An Example.

Find first four nonzero terms of y_1, y_2 of

$$e^x\, y'' + x\, y = 0 \tag{1}$$

Solution. We write

$$y = \sum_{n=0}^{\infty} a_n\, x^n \tag{2}$$

Substitute into equation we get

$$e^x \sum_{n=0}^{\infty} a_n\, x^n + x \sum_{n=0}^{\infty} a_n\, x^n = 0. \tag{3}$$

Now it is clear that we have to expand e^x too.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{4}$$

which makes the equation

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} a_n\, x^n\right)'' + x \sum_{n=0}^{\infty} a_n\, x^n = 0. \tag{5}$$

However, as there is in general no good way of writing simple formulas for coefficients of the result of a product of power series, we cannot expect to write down a simple recurrence relation. Realizing that all we need is four nonzero terms, we try to work things out in a more ad hoc way – writing down a few terms for each power series involved.

- Finding y_1. Set $a_0 = 1, a_1 = 0$. To get four nonzero terms, we have to compute at least up to a_4. The lowest order term in which a_4 appears is $1 \cdot 12\, a_4\, x^2$, so we have to balance the equation

  $$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} a_n\, x^n\right)'' + x \sum_{n=0}^{\infty} a_n\, x^n = 0. \tag{6}$$

  to at least x^2 term. To do this recall: To get correct coefficients of x^k in $\left(\sum_{n=0}^{\infty} a_n\, x^n\right)\left(\sum_{n=0}^{\infty} b_n\, x^n\right)$, we have to expand each series up to x^k. So we write

  $$\left(1 + x + \frac{x^2}{2} + \cdots\right)(2\, a_2 + 6\, a_3\, x + 12\, a_4\, x^2 + \cdots) + x + \cdots = 0 \tag{7}$$

  and conclude

  $$\begin{aligned} 2\, a_2 &= 0, & (8)\\ 2\, a_2 + 6\, a_3 + 1 &= 0, & (9)\\ a_2 + 6\, a_3 + 12\, a_4 &= 0. & (10) \end{aligned}$$

  This gives

  $$a_2 = 0, \quad a_3 = -\frac{1}{6}, \quad a_4 = \frac{1}{12}. \tag{11}$$

  We still need one more nonzero a_n. We compute a_5 by expanding the series one more term:

  $$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)(2\, a_2 + 6\, a_3\, x + 12\, a_4\, x^2 + 20\, a_5\, x^3 + \cdots) + x + a_2\, x^3 + \cdots = 0. \tag{12}$$

  The x^3 balance is (note that the $1, x, x^2$ balances have already been computed, x^3 is the only thing new):

  $$\frac{a_2}{3} + 3\, a_3 + 12\, a_4 + 20\, a_5 + a_2 = 0 \tag{13}$$

which gives $a_5 = -\frac{1}{40}$. As $a_5 \neq 0$ we have enough nonzero terms now:

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + \cdots \tag{14}$$

- Finding y_2. Setting $a_0 = 0$, $a_1 = 1$ we have

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots) + x^2 + a_2 x^3 + \cdots = 0. \tag{15}$$

Carrying out the multiplication, we have

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4 + 1)x^2 + \left(\frac{a_2}{3} + 3a_3 + 12a_4 + 20a_5 + a_2\right)x^3 + \cdots = 0. \tag{16}$$

The recurrence relations are

$$2a_2 = 0, \tag{17}$$

$$2a_2 + 6a_3 = 0, \tag{18}$$

$$a_2 + 6a_3 + 12a_4 + 1 = 0, \tag{19}$$

$$\frac{a_2}{3} + 3a_3 + 12a_4 + 20a_5 + a_2 = 0, \tag{20}$$

which give

$$a_2 = 0; \quad a_3 = 0; \quad a_4 = -\frac{1}{12}; \quad a_5 = \frac{1}{20}. \tag{21}$$

Thus

$$y_2 = x - \frac{1}{12}x^4 + \frac{1}{20}x^5 + \cdots \tag{22}$$

We only have 3 nonzero terms!

- Finding the 4th term.

  To find the 4th term, we need to expand everything to higher power. Let's try expanding to x^4:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\cdots\right)(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 \cdots) + x^2 + a_2 x^3 + a_3 x^4 \cdots = 0. \tag{23}$$

This gives a new recurrence relation via setting coefficients of x^4 to be 0:

$$\frac{a_2}{12} + a_3 + 6a_4 + 20a_5 + 30a_6 + a_3 = 0. \tag{24}$$

We obtain

$$a_6 = -\frac{1}{60}. \tag{25}$$

The updated y_2 is now

$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{20}x^5 - \frac{1}{60}x^6 + \cdots \tag{26}$$

Now we have 4 nonzero terms.

Remark 1. Note that y_1 solves the equation with initial values $y(0) = 1, y'(0) = 0$ and y_2 solves the equation with $y(0) = 0, y'(0) = 1$.

Regular points and singular points.

- Turns out that we can find out a **lower bound** of the radius of convergence for power series solutions **without** actually solving the equation. To do this we have to first write the equation into its standard form

$$y'' + p(x)y' + q(x)y = 0. \tag{27}$$

Theorem 2. *The radius of convergence ρ for the power series solution satisfies*

$$\rho \geqslant \min(\rho_1, \rho_2) \tag{28}$$

where ρ_1, ρ_2 is determined through

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n \qquad \text{for } |x - x_0| < \rho_1; \tag{29}$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n \qquad \text{for } |x - x_0| < \rho_2. \tag{30}$$

Remark 3. Note that ρ_1, ρ_2 **may not be** the radii of convergence for the Taylor expansion of p and q. For example, the Taylor expansion of $e^{-\frac{1}{x^2}}$ at $x_0 = 0$ has radius of convergence ∞, but the function equals its Taylor expansion only at x_0 and nowhere else.

- Often the following theorem is even easier to use:

Theorem 4. *The radius of convergence for the power series solution satisfies*

$$\rho \geqslant \text{distance of } x_0 \text{ to the nearest } \textbf{complex} \text{ singular point of the equation.} \tag{31}$$

To be able to apply this we need the following notions:

- A point is singular for the equation (**in standard form!**)

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{32}$$

  if either p or q (or both) is not analytic at at this point; Otherwise it's called "regular".

- A function $f(x)$ is analytic at a point x_0 if there is a sequence a_n and a number $\rho > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{33}$$

  holds $|x - x_0| < \rho$.

Remark 5. The function $e^{-\frac{1}{x^2}}$ is a typical example illustrating the following subtle fact: f is analytic at x_0 is **not the same** as "The Taylor expansion of f at x_0 has positive radius of convergence".

- How to tell?

From the above remark we see that it's not possible to tell whether $f(x)$ is analytic at a certain point x_0 from looking at its Taylor expansion. Then how to? In theory we have to do the following:

1. Compute its Taylor expansion at x_0;
2. Show

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{34}$$

   holds $|x - x_0| < \rho$ for some positive ρ.

The second step, of course is totally ad hoc and can be very difficult.

Fortunately that are several "rules of thumb" which are enough for this class.

1. $e^x, \sin x, \cos x$ and polynomials are analytic for all x; $\ln(1 + x)$ is analytic for $|x| < 1$.
2. If $f(x)$ is analytic at x_0 and $g(x)$ is analytic at $f(x_0)$, then the composite function $g(f(x))$ is analytic at x_0. For example, e^{x^2} is analytic everywhere.
3. If $f(x)$ and $g(x)$ are both analytic at x_0, then $f \pm g$ and $f g$ are analytic at x_0;
4. If $f(x)$ and $g(x)$ are analytic at x_0 and $g(x_0) \neq 0$, then $\frac{f}{g}$ is analytic at x_0.