LECTURE 15 UNDETERMINED COEFFICIENTS

10/14/2011

Review: Computing $(a+b\,i)^{1/n}$.

1. Write $a+b\,i=R\,e^{i(\theta_0+2k\pi)}$;
2. Write
$$(a+b\,i)^{1/n}=R^{1/n}\exp\left[i\,\frac{\theta_0+2\,k\,\pi}{n}\right]. \tag{1}$$
3. Set $k=$ n consecutive numbers (for example $0,1,\ldots,n-1$, or $-\frac{n}{2}+1,\ldots,0,\ldots,\frac{n}{2}$ when n is even and similarly when n is odd. Each value of k gives one root.
4. Simplify if possible.

Undetermined Coefficients.

Undetermined coefficients for higher order equations follow the same rule as for 2nd order equations.

1. Solve the homogeneous equation. Get the list of roots $r_1,\ldots,r_n$ for the characteristic equation.
2. Check $g(t)$:
   - If $g(t)=e^{rt}\,(a_0+\cdots+a_n\,t^n)$, then
   $$y_p=t^s\,e^{rt}\,(A_0+\cdots+A_n\,t^n); \tag{2}$$
   $s=\#$ of times r appears in the list $r_1,\ldots,r_n$.
   - If $g(t)=e^{\alpha t}\cos\beta t\,(a_0+\cdots+a_n\,t^n)+e^{\alpha t}\sin\beta t\,(b_0+\cdots+b_m\,t^m)$, then
   $$y_p=t^s\,e^{\alpha t}\cos\beta t\,(A_0+\cdots+A_k\,t^k)+t^s\,e^{\alpha t}\sin\beta t\,(B_0+\cdots+B_k\,t^k) \tag{3}$$
   $s=\#$ of times $\alpha+\beta\,i$ appears in the list $r_1,\ldots,r_n$. Here k equals the larger of n,m.
   - If $g(t)$ is not of either type but can be written as $g_1+\cdots+g_l$ with each g_i falling in one of the above two types, then obtain y_{p_i} for each g_i and write
   $$y_p=y_{p1}+\cdots+y_{pl}. \tag{4}$$
3. Substitute y_p obtained above into the equation to fix all the coefficients.
4. Write down the general solution. $y=C_1\,y_1+\cdots+C_n\,y_n+y_p$.
5. If initial value problem, use initial conditions to get $C_1,\ldots,C_n$.

Example 1. Determine the form of y_p for
$$y^{(4)}-2\,y''+y=e^t+\sin t. \tag{5}$$

Solution.

First solve the homogeneous equation:
$$y^{(4)}-2\,y''+y=0 \tag{6}$$

Characteristic equation:
$$r^4-2\,r^2+1=0\Longrightarrow r_{1,2,3,4}=1,1,-1,-1. \tag{7}$$

Next look at g: $e^t+\sin t$. We see that it's not in either type. But e^t and $\sin t$ are.
$$e^t=e^{1\cdot t}\,(1)\Longrightarrow y_{p1}=t^s\,e^t\,A \tag{8}$$

As 1 appears in the list of roots twice, $s=2$. So $y_{p1}=A\,t^2\,e^t$. Next

$$\sin t = e^{0\cdot t}\sin 1\cdot t\,(1) \Longrightarrow y_{p2} = t^s\,e^{0\cdot t}\cos 1\cdot t\,B + t^s\,e^{0\cdot t}\sin 1\cdot t\,C. \tag{9}$$

As $0+1\cdot i=i$ does not appear in the list of roots, $s=0$. So $y_{p2}=B\cos t+C\sin t$.

Therefore

$$y_p = A\,t^2\,e^t + B\cos t + C\sin t \tag{10}$$

Example 2. Solve

$$y^{(4)} + 2\,y''' + y'' + 8\,y' - 12\,y = e^{-t}; \qquad y(0)=3, y'(0)=0, y''(0)=-1, y'''(0)=2. \tag{11}$$

Solution.

First solve the homogeneous equation:

$$y^{(4)} + 2\,y''' + y'' + 8\,y' - 12\,y = 0 \tag{12}$$

Characteristic equation:

$$r^4 + 2\,r^3 + r^2 + 8\,r - 12 = 0. \tag{13}$$

Inspection gives $r_1=1$. Factorize

$$r^4 + 2\,r^3 + r^2 + 8\,r - 12 = (r-1)\,(r^3 + 3\,r^2 + 4\,r + 12). \tag{14}$$

Thus the other three roots comes from solving

$$r^3 + 3\,r^2 + 4\,r + 12 = 0. \tag{15}$$

Inspection gives $r_2=-3$. Factorize

$$r^3 + 3\,r^2 + 4\,r + 12 = (r+3)\,(r^2+4). \tag{16}$$

The last two roots solve $r^2+4=0$ so we finally get

$$r_{1,2,3,4} = 1, -3, \pm 2\,i. \tag{17}$$

And the general solution for the homogeneous equation is

$$C_1\,e^t + C_2\,e^{-3t} + C_3\cos 2\,t + C_4\sin 2\,t. \tag{18}$$

Next we guess y_p. As

$$g(t) = e^{-t} \tag{19}$$

we have

$$y_p = t^s\,e^{-t}\,A. \tag{20}$$

Check: -1 appears 0 times in the roots list so $s=0$.

Substitute $y_p = A\,e^{-t}$ into the equation we have

$$e^{-t} = A\,e^{-t} - 2\,A\,e^{-t} + A\,e^{-t} - 8\,A\,e^{-t} - 12\,A\,e^{-t} = -20\,A\,e^{-t}. \tag{21}$$

So $A=-\frac{1}{20}$ and $y_p=-\frac{1}{20}\,e^{-t}$.

The general solution is then

$$y = C_1\,e^t + C_2\,e^{-3t} + C_3\cos 2\,t + C_4\sin 2\,t - \frac{1}{20}\,e^{-t}. \tag{22}$$

To fix $C_1, \ldots, C_4$ using the initial conditions we have to first prepare:

$$y' = C_1\,e^t - 3\,C_2\,e^{-3t} - 2\,C_3\sin 2\,t + 2\,C_4\cos 2\,t + \frac{1}{20}\,e^{-t}; \tag{23}$$

$$y'' = C_1\,e^t + 9\,C_2\,e^{-3t} - 4\,C_3\cos 2\,t - 4\,C_4\sin 2\,t - \frac{1}{20}\,e^{-t}; \tag{24}$$

$$y''' = C_1\,e^t - 27\,C_2\,e^{-3t} + 8\,C_3\sin 2\,t - 8\,C_4\cos 2\,t + \frac{1}{20}\,e^{-t}. \tag{25}$$

Therefore

$$y(0)=3 \Longrightarrow C_1+C_2+C_3=3+\frac{1}{20}=\frac{61}{20}; \tag{26}$$

$$y'(0)=0 \Longrightarrow C_1-3\,C_2+2\,C_4=-\frac{1}{20}; \tag{27}$$

$$y''(0)=-1 \Longrightarrow C_1+9\,C_2-4\,C_3=-\frac{19}{20}; \tag{28}$$

$$y'''(0)=2 \Longrightarrow C_1-27\,C_2-8\,C_4=\frac{39}{20}. \tag{29}$$

We apply Gaussian elimination to solve this system. (Let $R1, R2$ denote row 1 row 2 etc.)

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \frac{61}{20} \\ 1 & -3 & 0 & 2 & -\frac{1}{20} \\ 1 & 9 & -4 & 0 & -\frac{19}{20} \\ 1 & -27 & 0 & -8 & \frac{39}{20} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & \frac{61}{20} \\ 0 & -4 & -1 & 2 & -\frac{31}{10} \\ 0 & 8 & -5 & 0 & -4 \\ 0 & -28 & -1 & -8 & -\frac{11}{10} \end{pmatrix} \tag{30}$$

$R2 \leftarrow R_2-R1; R3 \leftarrow R_3-R_1; R_4 \leftarrow R_4-R1$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \frac{61}{20} \\ 0 & -4 & -1 & 2 & -\frac{31}{10} \\ 0 & 0 & -7 & 4 & -\frac{51}{5} \\ 0 & 0 & 6 & -22 & \frac{103}{5} \end{pmatrix} \tag{31}$$

$R3 \leftarrow R_3+2\times R2; R_4 \leftarrow R_4-7\times R2$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \frac{61}{20} \\ 0 & -4 & -1 & 2 & -\frac{31}{10} \\ 0 & 0 & -7 & 4 & -\frac{71}{5} \\ 0 & 0 & 0 & -\frac{130}{7} & \frac{83}{7} \end{pmatrix} \tag{32}$$

$R_4 \leftarrow R_4+\frac{6}{7}\times R3.$

Thus $C_1, \ldots, C_4$ satisfy

$$C_1+C_2+C_3 = \frac{61}{20} \tag{33}$$

$$-4\,C_2-C_3+2\,C_4 = -\frac{31}{10} \tag{34}$$

$$-7\,C_3+4\,C_4 = -\frac{71}{5} \tag{35}$$

$$-\frac{130}{7}\,C_4 = \frac{83}{7}. \tag{36}$$

and can be solved one by one $C_4 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1$:

$$C_4 = -\frac{83}{130}; \tag{37}$$

$$C_3 = \frac{757}{5\times 13\times 7}; \tag{38}$$

$$C_2 = \frac{29}{13\times 7\times 8}; \tag{39}$$

$$C_1 = \frac{4901}{13\times 7\times 8\times 5}; \tag{40}$$

The final answer is then

$$y=\frac{4901}{13\times 7\times 8\times 5}\,e^t+\frac{29}{13\times 7\times 8}\,e^{-3t}+\frac{757}{5\times 13\times 7}\cos 2\,t-\frac{83}{130}\sin 2\,t-\frac{1}{20}\,e^{-t}. \tag{41}$$

(Sorry for not foreseeing this! But rest assured: Such evil numbers will not appear in exams!)