Lecture 13 Higher Order Linear Equations

10/05/2011

Theory.

$$P_0(t)\,\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + P_1(t)\,\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + P_n(t)\,y = G(t). \tag{1}$$

or standard form

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + p_1(t)\,\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + p_n(t)\,y = g(t). \tag{2}$$

Usually simply

$$y^{(n)} + p_1(t)\,y^{(n-1)} + \cdots + p_n(t)\,y = g(t). \tag{3}$$

- General solution

  $$y = C_1\,y_1 + \cdots + C_n\,y^n + y_p \tag{4}$$

  with $y_1, \ldots, y_n$ fundamental set of the homogeneous equation[1]

  $$y^{(n)} + p_1(t)\,y^{(n-1)} + \cdots + p_n(t)\,y = 0. \tag{6}$$

  and "particular solution" y_p solves the non-homogeneous equation itself.

- To check linear independence, use Wronskian

  $$W[y_1, \ldots, y_n] = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}. \tag{7}$$

  When $y_1, \ldots, y_n$ solves the same homogeneous equation, they are linearly independent if and only if $W \neq 0$ at some point t_0.

- Calculation of determinants.

  - $n = 2, 3$: formulas.

    $$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}. \tag{8}$$

    $$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{21}\,a_{32}\,a_{13} - a_{13}\,a_{22}\,a_{31} - a_{21}\,a_{12}\,a_{33} - a_{23}\,a_{32}\,a_{11}. \tag{9}$$

  - For all n: Co-factor expansion.

    $$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{\mathrm{nn}} \end{pmatrix} = \sum_{j=1}^{n} a_{kj}\,(-1)^{k+j} \det(A_{kj}) = \sum_{i=1}^{n} a_{ik} (-1)^{i+k} \det(A_{ik}). \tag{10}$$

    for any $k = 1, \ldots, n$. Here the matrix A_{kj} is the $(n-1) \times (n-1)$ matrix obtained from the original matrix by deleting the k-th row and the j-th column (that is deleting the row and the column containing a_{kj}). For example, A_{13} for the 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ can be obtained as follows:

    $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Longrightarrow A_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}. \tag{11}$$

1. Are solutions; Are linearly independent:

$$C_1\,y_1 + \cdots + C_n\,y_n = 0 \Longrightarrow C_1 = \cdots = C_n = 0. \tag{5}$$

We can derive the 2×2 formula from $\det(a) = a$ if a is a number using the co-factor expansion: Notice that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Longrightarrow A_{11} = (\ a_{22}\ ); \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Longrightarrow A_{21} = (\ a_{12}\ ). \tag{12}$$

So

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\,(-1)^{1+1} \det A_{11} + a_{21}\,(-1)^{2+1} \det A_{21} = a_{11}\,a_{22} - a_{21}\,a_{12}. \tag{13}$$

If you have time, try deriving the 3×3 formula from the 2×2 formula using co-factor expansion.

- ○ Gaussian elimination method. This is in fact the most efficient method but we don't have time to fully discuss it[2] and furthermore for 2×2 and 3×3 matrices it is not much faster than the formula method anyway, especially when entries of the matrix are functions.

Solving constant-coefficient, homogeneous, linear equations.

$$a_0\, y^{(n)} + \cdots + a_n\, y = 0. \tag{14}$$

1. Solve the characteristic equation

$$a_0\, r^n + \cdots + a_n = 0 \tag{15}$$

   and get $r_1, r_2, \ldots, r_n$.

2. Write down $y_1, \ldots, y_n$.
3. $y = C_1\, y_1 + \cdots + C_n\, y_n$.

- Examples of step 2. Obtain y from roots.
  - ○ $1, 2, 3, 4$. Distinct real roots:

$$y = C_1\, e^t + C_2\, e^{2t} + C_3\, e^{3t} + C_4\, e^{4t}. \tag{16}$$

  - ○ $1, 1, 1, 1, 3$.
    Rule: Repeated real roots (k times) gives $e^{rt}, t\, e^{rt}, \ldots, t^{k-1}\, e^{rt}$. So

$$y = C_1\, e^t + C_2\, t\, e^t + C_3\, t^2\, e^t + C_4\, t^3\, e^t + C_5\, e^{3t}. \tag{17}$$

  - ○ $2+i, 2-i, 2+i, 2-i, 3$.
    Rule: Repeated pairs of complex roots gives (if we have $\alpha \pm \beta i$ repeated k times)

$$e^{\alpha t} \cos\beta t, e^{\alpha t} \sin \beta t, \ldots, t^{k-1}\, e^{\alpha t} \cos \beta t, t^{k-1}\, e^{\alpha t} \sin \beta t. \tag{18}$$

    So

$$y = C_1\, e^{2t} \cos t + C_2\, e^{2t} \sin t + C_3\, t\, e^{2t} \cos t + C_4\, t\, e^{2t} \sin t + C_5\, e^{3t}. \tag{19}$$

- Solvable equations. In our class basically there are two types of equations:
  - ○ Simple equations. That is those whose characteristic equation can be solved through repeating
    1. Guess a root. Usually the first guess is 1. If doesn't work, -1. Then $2, -2, \ldots$ until giving up.
    2. Factorize.

    Example 1. Solve

$$y''' - y'' + y' - y = 0. \tag{20}$$

    Characteristic equation:

$$r^3 - r^2 + r - 1 = 0. \tag{21}$$

    Try $r = 1$: $1 - 1 + 1 - 1 = 0$ so $r_1 = 1$. Factorize

$$r^3 - r^2 + r - 1 = (r-1)\,(r^2 + 1). \tag{22}$$

2. Because such discussion will have to involve a detailed discussion of properties of determinants.

So the remaining roots are those of $r^2+1=0$. Therefore $r_{2,3}=\pm i$. So

$$y = C_1 e^t + C_2 \cos t + C_3 \sin t. \tag{23}$$

- Special equations. Equestions like $y^{(5)} - y = 0$.
  Will discuss next time.