Lecture 11 Undetermined Coefficients (cont.)

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The method: Review.

- Target equations:

$$a\,y'' + b\,y' + c\,y = g(t) \tag{1}$$

  with a, b, c constants, and $g(t)$ of one of the following two types

  1. $g(t) = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n)$;
  2. $g(t) = e^{\alpha t}\cos\beta t\,(a_0 + \cdots + a_n\,t^n)$ or $g(t) = e^{\alpha t}\sin\beta t\,(a_0 + \cdots + a_n\,t^n)$.

- Procedure.

  - Step 1: Solve the homogeneous equation

$$a\,y'' + b\,y' + c\,y = 0. \tag{2}$$

    Get y_1, y_2 and also a list of roots for the characteristic equation $a\,r^2 + b\,r + c = 0$.

  - Step 2: Guess a "particular solution" y_p using the following rules:

    – If $g(t) = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n)$, guess

$$y_p = t^s\,e^{\alpha t}\,(A_0 + \cdots + A_n\,t^n). \tag{3}$$

      with s determined as follows:

      - $s = 0$ if α is not a root to the characteristic equation.
      - $s = 1$ if α is a single root;
      - $s = 2$ if α is a repeated root (double root).

      Here $A_0, \ldots, A_n$ are the "undetermined coefficients" that need to be fixed.

    – If $g(t) = e^{\alpha t}\cos\beta t\,(a_0 + \cdots + a_n\,t^n)$ or $g(t) = e^{\alpha t}\sin\beta t\,(a_0 + \cdots + a_n\,t^n)$, guess

$$y_p = t^s\,e^{\alpha t}\cos\beta t\,(A_0 + \cdots + A_n\,t^n) + t^s\,e^{\alpha t}\sin\beta t\,(B_0 + \cdots + B_n\,t^n). \tag{4}$$

      with s determined as follows:

      - $s = 0$ if $\alpha + i\,\beta$ is not a root to the characteristic equation.
      - $s = 1$ if $\alpha + i\,\beta$ is a root to the characteristic equation.

      Here $A_0, \ldots, A_n, B_0, \ldots, B_n$ are the "undetermined coefficients" that need to be fixed.

    Then substitute y_p into the equation $a\,y'' + b\,y' + c\,y = g(t)$ to find the coefficients.

  - Step 3: Write down the solution.

$$y = C_1\,y_1 + C_2\,y_2 + y_p. \tag{5}$$

- Some remarks.

  - What if $g(t)$ is of neither form? Check whether $g(t)$ can be written as a sum of g_i's while each g_i falls into one of the two types above, and find y_{pi} for each g_i. y_p is then the sum of all the y_{pi}'s.

    For example, if $g(t) = \cos t + t^2 + 3\sin 2t$. We see that we can set $g_1 = \cos t$, $g_2 = t^2$, $g_3 = 3\sin 2t$.

    Then we can find y_p through:

    – $g_1 = \cos t \Longrightarrow y_{p1} = t^s\,[A\cos t + B\sin t]$. Find A, B to get y_{p1}.[1]

- $g_2 = t^2 \Longrightarrow y_{p2} = t^s\,[A_0 + A_1\,t + A_2\,t^2]$. Find A_0, A_1, A_2 to get y_{p2}.
- $g_3 = 3\sin 2\,t \Longrightarrow y_{p3} = t^s\,[A\cos 2\,t + B\sin 2\,t]$. Find A, B to get y_{p3}.

Now $y_p = y_{p1} + y_{p2} + y_{p3}$.

- When $g(t) = e^{\alpha t}\cos\beta t\,(a_0 + \cdots + a_n\,t^n) + e^{\alpha t}\sin\beta t\,(b_0 + \cdots + b_m\,t^m)$, the above procedure gives

$$y_{p1} = e^{\alpha t}\cos\beta t\,(A_0 + \cdots + A_n\,t^n) + e^{\alpha t}\sin\beta t\,(B_0 + \cdots + B_n\,t^n) \tag{6}$$

and

$$y_{p2} = e^{\alpha t}\cos\beta t\,(A_0' + \cdots + A_m'\,t^m) + e^{\alpha t}\sin\beta t\,(B_0' + \cdots + B_m'\,t^m). \tag{7}$$

We see that we can simplify things a little bit by writing directly

$$y_p = e^{\alpha t}\cos\beta t\,(A_0 + \cdots + A_k\,t^k) + e^{\alpha t}\sin\beta t\,(B_0 + \cdots + B_k\,t^k) \tag{8}$$

and try to find out the coefficients $A_0, \ldots, A_k, B_0, \ldots, B_k$. Here $k = \max\{m, n\}$ is the bigger one of m and n. For example if $m = 3, n = 2$ then $k = 3$.

Why does this method work? Some explanations.

- Why is the general solution

$$y = C_1\,y_1 + C_2\,y_2 + y_p? \tag{9}$$

Notice that y is in fact the sum of *the general solution for* $a\,y'' + b\,y' + c\,y = 0$ and *one single solution for* $a\,y'' + b\,y' + c\,y = g(t)$.

To understand why, think as follows. Consider the equation

$$a\,y'' + b\,y' + c\,y = g(t). \tag{10}$$

Now let y_p be one solution of this equation. Let's see what equation should $y - y_p$ satisfy. Try

$$a\,(y - y_p)'' + b\,(y - y_p)' + c\,(y - y_p) = a\,y'' + b\,y' + c\,y - a\,y_p'' - b\,y_p' - c\,y_p = g(t) - g(t) = 0. \tag{11}$$

So

The difference $y - y_p$ satisfies the homogeneous equation!

In other words, any solution y is a sum of y_p and a solution to the homogeneous equation $a\,y'' + b\,y' + c\,y = 0$. But all solutions to this equation can be written as $C_1\,y_1 + C_2\,y_2$. So the general solution to

$$a\,y'' + b\,y' + c\,y = g(t). \tag{12}$$

must be

$$y = C_1\,y_1 + C_2\,y_2 + y_p. \tag{13}$$

- Why guess
  - If $g(t) = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n)$, guess

$$y_p = t^s\,e^{\alpha t}\,(A_0 + \cdots + A_n\,t^n). \tag{14}$$

with s determined as follows:

- $s = 0$ if α is not a root to the characteristic equation.
- $s = 1$ if α is a single root;
- $s = 2$ if α is a repeated root (double root).

Here $A_0, \ldots, A_n$ are the "undetermined coefficients" that need to be fixed.

1. The equation is not given here so we cannot fix s.

- If $g(t)=e^{\alpha t}\cos\beta t\,(a_0+\cdots+a_n t^n)$ or $g(t)=e^{\alpha t}\sin\beta t\,(a_0+\cdots+a_n t^n)$, guess

$$y_p=t^s e^{\alpha t}\cos\beta t\,(A_0+\cdots+A_n t^n)+t^s e^{\alpha t}\sin\beta t\,(B_0+\cdots+B_n t^n). \tag{15}$$

  with s determined as follows:

  - $s=0$ if $\alpha+i\beta$ is not a root to the characteristic equation.
  - $s=1$ if $\alpha+i\beta$ is a root to the characteristic equation.

  Here $A_0,\ldots,A_n,B_0,\ldots,B_n$ are the "undetermined coefficients" that need to be fixed.

It does not have simple explanation. But some understanding can be obtained through the following. Consider the problem

$$a\,y''+b\,y'+c\,y=e^{\alpha t}. \tag{16}$$

We need one solution y_p, a function, some linear combinations of whose derivatives is $e^{\alpha t}$. But we know that to get $e^{\alpha t}$ through differentiation, basically we have to start from something like $A\,e^{\alpha t}$. So the guess $y_p=A\,e^{\alpha t}$.

One more example.

Example 1. Solve

$$y''+2\,y'+5\,y=4\,e^{-t}\cos 2\,t. \tag{17}$$

Solution.

- First solve the homogeneous equation

$$y''+2\,y'+5\,y=0 \tag{18}$$

  whose characteristic equation is $r^2+2\,r+5=0$. This gives

$$r_{1,2}=\frac{-2\pm\sqrt{2^2-4\cdot 5}}{2}=-1\pm 2\,i. \tag{19}$$

  So

$$y_1=e^{-t}\cos 2\,t,\qquad y_2=e^{-t}\sin 2\,t. \tag{20}$$

- Next determine the form of y_p. We observe

$$g(t)=4\,e^{-t}\cos 2\,t=e^{\alpha t}\cos\beta t\,(a_0+\cdots+a_n t^n) \tag{21}$$

  with $\alpha=-1,\beta=2,n=0$. So

$$y_p=t^s e^{\alpha t}\cos\beta t\,(A_0+\cdots+A_n t^n)+t^s e^{\alpha t}\sin\beta t\,(B_0+\cdots+B_n t^n). \tag{22}$$

  As $\alpha+i\beta=-1+2\,i$ is a solution to the characteristic equation $r^2+2\,r+5=0$, we need to set $s=1$. So our guess is

$$y_p=t\,e^{-t}\,[A_0\cos 2\,t+B_0\sin 2\,t]. \tag{23}$$

- Substitute y_p back into the equation.

  - Preparation. We compute

$$\begin{aligned} y_p' &= (t\,e^{-t}\,[A_0\cos 2\,t+B_0\sin 2\,t])' \\ & \quad \text{Use } (f\,g\,h)'=f'\,g\,h+f\,g'\,h+f\,g\,h' \\ &= e^{-t}\,(A_0\cos 2\,t+B_0\sin 2\,t)-t\,e^{-t}\,(A_0\cos 2\,t+B_0\sin 2\,t) \\ & \quad +t\,e^{-t}\,(-2\,A_0\sin 2\,t+2\,B_0\cos 2\,t) \\ &= e^{-t}\,(A_0\cos 2\,t+B_0\sin 2\,t)+t\,e^{-t}\,((2\,B_0-A_0)\cos 2\,t+(-2\,A_0-B_0)\sin 2\,t). \end{aligned} \tag{24}$$

and

$$\begin{aligned} y_p'' &= [e^{-t}(A_0\cos 2t + B_0\sin 2t) + t\,e^{-t}((2B_0 - A_0)\cos 2t + (-2A_0 - B_0)\sin 2t)]' \\ &= [e^{-t}(A_0\cos 2t + B_0\sin 2t)]' + [t\,e^{-t}((2B_0 - A_0)\cos 2t + (-2A_0 - B_0)\sin 2t)]' \\ &= -e^{-t}(A_0\cos 2t + B_0\sin 2t) + e^{-t}(-2A_0\sin 2t + 2B_0\cos 2t) \\ &\quad + e^{-t}((2B_0 - A_0)\cos 2t + (-2A_0 - B_0)\sin 2t) \\ &\quad - t\,e^{-t}((2B_0 - A_0)\cos 2t + (-2A_0 - B_0)\sin 2t) \\ &\quad + t\,e^{-t}[-2(2B_0 - A_0)\sin 2t + 2(-2A_0 - B_0)\cos 2t] \\ &= (-A_0 + 2B_0 + (2B_0 - A_0))\,e^{-t}\cos 2t \\ &\quad + (-B_0 - 2A_0 + (-2A_0 - B_0))\,e^{-t}\sin 2t \\ &\quad + (-(2B_0 - A_0) + 2(-2A_0 - B_0))\,t\,e^{-t}\cos 2t \\ &\quad + (-(-2A_0 - B_0) - 2(2B_0 - A_0))\,t\,e^{-t}\sin 2t \\ &= (4B_0 - 2A_0)\,e^{-t}\cos 2t + (-4A_0 - 2B_0)\,e^{-t}\sin 2t \\ &\quad + (-3A_0 - 4B_0)\,t\,e^{-t}\cos 2t + (4A_0 - 3B_0)\,t\,e^{-t}\sin 2t. \end{aligned}$$

- Substitute into the equation:

$$\begin{aligned} 4e^{-t}\cos 2t &= y_p'' + 2y_p' + 5y_p \\ &= (4B_0 - 2A_0)\,e^{-t}\cos 2t + (-4A_0 - 2B_0)\,e^{-t}\sin 2t \\ &\quad + (-3A_0 - 4B_0)\,t\,e^{-t}\cos 2t + (4A_0 - 3B_0)\,t\,e^{-t}\sin 2t \\ &\quad + 2\{e^{-t}(A_0\cos 2t + B_0\sin 2t) + t\,e^{-t}((2B_0 - A_0)\cos 2t + (-2A_0 - B_0)\sin 2t)\} \\ &\quad + 5t\,e^{-t}[A_0\cos 2t + B_0\sin 2t] \\ &= (4B_0 - 2A_0 + 2A_0)\,e^{-t}\cos 2t \\ &\quad + (-4A_0 - 2B_0 + 2B_0)\,e^{-t}\sin 2t \\ &\quad + (-3A_0 - 4B_0 + 4B_0 - 2A_0 + 5A_0)\,t\,e^{-t}\cos 2t \\ &\quad + (4A_0 - 3B_0 - 4A_0 - 2B_0 + 5B_0)\,t\,e^{-t}\sin 2t \\ &= 4B_0\,e^{-t}\cos 2t - 4A_0\,e^{-t}\sin 2t. \end{aligned}$$

Therefore

$$B_0 = 1, A_0 = 0 \Longrightarrow y_p = t\,e^{-t}\sin 2t. \tag{25}$$

- Check y_p if time allows:

$$y_p' = e^{-t}\sin 2t - t\,e^{-t}\sin 2t + 2t\,e^{-t}\cos 2t; \tag{26}$$

$$y_p'' = -e^{-t}\sin 2t + 2e^{-t}\cos 2t - e^{-t}\sin 2t + t\,e^{-t}\sin 2t - 2t\,e^{-t}\cos 2t + 2e^{-t}\cos 2t - 2t\,e^{-t}\cos 2t - 4t\,e^{-t}\sin 2t = 4e^{-t}\cos 2t - 2e^{-t}\sin 2t - 4t\,e^{-t}\cos 2t - 3t\,e^{-t}\sin 2t. \tag{27}$$

So

$$\begin{aligned} y_p'' + 2y_p' + 5y_p &= 4e^{-t}\cos 2t - 2e^{-t}\sin 2t - 4t\,e^{-t}\cos 2t - 3t\,e^{-t}\sin 2t \\ &\quad + 2e^{-t}\sin 2t - 2t\,e^{-t}\sin 2t + 4t\,e^{-t}\cos 2t \\ &\quad + 5t\,e^{-t}\sin 2t \\ &= 4e^{-t}\cos 2t. \end{aligned} \tag{28}$$

Hooray...

- Finally write down the general solution

$$y = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + t\,e^{-t}\sin 2t. \tag{29}$$

Sorry I should have used this as a homework problem instead of an in-class example...and many thanks to those who helped me during today's lecture...