Lecture 10 Undetermined Coefficients

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The method.

- Target equations:

$$a\,y'' + b\,y' + c\,y = g(t) \tag{1}$$

  with a, b, c constants, and $g(t)$ of one of the following two types

  1. $g(t) = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n)$;
  2. $g(t) = e^{\alpha t}\cos\beta t\,(a_0 + \cdots + a_n\,t^n)$ or $g(t) = e^{\alpha t}\sin\beta t\,(a_0 + \cdots + a_n\,t^n)$.

- Procedure.

  ◦ Step 1: Solve the homogeneous equation

$$a\,y'' + b\,y' + c\,y = 0. \tag{2}$$

  Get y_1, y_2 and also a list of roots for the characteristic equation $a\,r^2 + b\,r + c = 0$.

  ◦ Step 2: Guess a "particular solution" y_p using the following rules:

  – If $g(t) = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n)$, guess

$$y_p = t^s\,e^{\alpha t}\,(A_0 + \cdots + A_n\,t^n). \tag{3}$$

  with s determined as follows:

  - $s = 0$ if α is not a root to the characteristic equation.
  - $s = 1$ if α is a single root;
  - $s = 2$ if α is a repeated root (double root).

  Here $A_0, \ldots, A_n$ are the "undetermined coefficients" that need to be fixed.

  – If $g(t) = e^{\alpha t}\cos\beta t\,(a_0 + \cdots + a_n\,t^n)$ or $g(t) = e^{\alpha t}\sin\beta t\,(a_0 + \cdots + a_n\,t^n)$, guess

$$y_p = t^s\,e^{\alpha t}\cos\beta t\,(A_0 + \cdots + A_n\,t^n) + t^s\,e^{\alpha t}\sin\beta t\,(B_0 + \cdots + B_n\,t^n). \tag{4}$$

  with s determined as follows:

  - $s = 0$ if $\alpha + i\,\beta$ is not a root to the characteristic equation.
  - $s = 1$ if $\alpha + i\,\beta$ is a root to the characteristic equation.

  Here $A_0, \ldots, A_n, B_0, \ldots, B_n$ are the "undetermined coefficients" that need to be fixed.

  Then substitute y_p into the equation $a\,y'' + b\,y' + c\,y = g(t)$ to find the coefficients.

  ◦ Step 3: Write down the solution.

$$y = C_1\,y_1 + C_2\,y_2 + y_p. \tag{5}$$

Examples.

Example 1. Solve

$$y'' - 2\,y' - 3\,y = 3\,e^{2t}. \tag{6}$$

Solution.

- Step 1. Solve $y'' - 2\,y' - 3\,y = 0$. Characteristic equation: $r^2 - 2\,r - 3 = 0$ gives $r_1 = 3, r_2 = -1$. So $y_1 = e^{3t}, y_2 = e^{-t}$.
- Step 2. We have

$$g(t) = 3\,e^{2t} = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n) \tag{7}$$

with $\alpha=2, n=0, a_0=3$. So guess

$$y_p = t^s\, e^{2t}\, A_0. \tag{8}$$

We further notice that $\alpha=2$ is not a root to the characteristic equation, so $s=0$. Thus our final guess is

$$y_p = A_0\, e^{2t}. \tag{9}$$

Substitute into equation

$$(A_0\, e^{2t})'' - 2\,(A_0\, e^{2t})' - 3\,(A_0\, e^{2t}) = 3\, e^{2t}. \tag{10}$$

Simplify:

$$4\, A_0\, e^{2t} - 4\, A_0\, e^{2t} - 3\, A_0\, e^{2t} = 3\, e^{2t}. \tag{11}$$

So $A_0 = -1$, which means

$$y_p = -e^{2t}. \tag{12}$$

Note. It's a good idea to check solution at this stage (sorry didn't do this in lecture!):

$$(-e^{2t})'' - 2\,(-e^{2t})' - 3\,(-e^{2t}) = -4\, e^{2t} + 4\, e^{2t} + 3\, e^{2t} = 3\, e^{2t}. \tag{13}$$

- Step 3. Write

$$y = C_1\, e^{3t} + C_2\, e^{-t} - e^{2t}. \tag{14}$$

Example 2. Solve

$$2\, y'' + 3\, y' + y = t^2 + 3 \sin t. \tag{15}$$

Solution.

- Step 0. Notice that $g(t) = t^2 + 3 \sin t$ satisfies neither $g(t) = e^{\alpha t}\, (a_0 + \cdots + a_n\, t^n)$ nor $g(t) = e^{\alpha t} \cos \beta t\, (a_0 + \cdots + a_n\, t^n)$ or $g(t) = e^{\alpha t} \sin \beta t\, (a_0 + \cdots + a_n\, t^n)$. However, we can break g into a sum of two functions $g_1 + g_2$ with

$$g_1 = t^2, \qquad g_2 = 3 \sin t. \tag{16}$$

Now it is clear that

  - $g_1 = e^{\alpha t}\, (a_0 + \cdots + a_n\, t^n)$ with $\alpha = 0, n = 2$, and
  - $g_2 = e^{\alpha t} \sin t\, (a_0 + \cdots + a_n\, t^n)$ with $\alpha = 0, \beta = 1, n = 0$.

It turns out that we can find the general solution for (note that the homogeneous part $2\, y'' + 3\, y' + y = 0$ is the same for both equations and therefore they share y_1, y_2.

$$2\, y'' + 3\, y' + y = t^2 \Longrightarrow y = C_1\, y_1 + C_2\, y_2 + y_{p1} \tag{17}$$

and

$$2\, y'' + 3\, y' + y = 3 \sin t \Longrightarrow y = C_1'\, y_1 + C_2'\, y_2 + y_{p2} \tag{18}$$

and then add them up to get the solution for

$$2\, y'' + 3\, y' + y = t^2 + 3 \sin t. \tag{19}$$

as

$$y = (C_1 + C_1')\, y_1 + (C_2 + C_2')\, y_2 + y_{p1} + y_{p2}. \tag{20}$$

But C_1, C_2, C_1', C_2' are just arbitrary constants, so the above is the same as

$$y = C_1\, y_1 + C_2\, y_2 + (y_{p1} + y_{p2}). \tag{21}$$

Effectively, we are saying $y_p = y_{p1} + y_{p2}$.

- Step 1. Solve homogeneous equation $2\, y'' + 3\, y' + y = 0$. Characteristic equation: $2\, r^2 + 3\, r + 1 = 0$. So

$$r_1 = -1/2, \qquad r_2 = -1, \qquad y_1 = e^{-t/2}, \qquad y_2 = e^{-t}. \tag{22}$$

- Step 2a. Find y_{p1}.

As $g_1 = e^{\alpha t}(a_0 + \cdots + a_n t^n)$ with $\alpha = 0, n = 2$ we guess

$$y_{p1} = t^s e^{\alpha t}(A_0 + \cdots + A_n t^n) \tag{23}$$

with $\alpha = 0, n = 2$. Furthermore as $\alpha = 0$ is not a root of the characteristic equation, we set $s = 0$ and consequently

$$y_p = A_0 + A_1 t + A_2 t^2. \tag{24}$$

Substitute into the equation:

$$2(A_0 + A_1 t + A_2 t^2)'' + 3(A_0 + A_1 t + A_2 t^2) + (A_0 + A_1 t + A_2 t^2) = t^2. \tag{25}$$

This is simply

$$4 A_2 + 3(A_1 + 2 A_2 t) + (A_0 + A_1 t + A_2 t^2) = t^2 \tag{26}$$

which in turn can be written as

$$(4 A_2 + 3 A_1 + A_0) + (6 A_2 + A_1) t + A_2 t^2 = t^2 = 0 + 0 \cdot t + 1 \cdot t^2. \tag{27}$$

We know that if two polynomials are equal:

$$a_0 + \cdots + a_n t^n = b_0 + \cdots + b_n t^n \tag{28}$$

then necessarily $a_0 = b_0, a_1 = b_1, \ldots, a_n = b_n$.

Therefore the "undetermined coefficients" must satisfy

$$4 A_2 + 3 A_1 + A_0 = 0 \tag{29}$$
$$6 A_2 + A_1 = 0 \tag{30}$$
$$A_2 = 1 \tag{31}$$

This can be solved in the order $A_2 \rightarrow A_1 \rightarrow A_0$ to get

$$A_2 = 1, \quad A_1 = -6, \quad A_0 = 14. \tag{32}$$

Thus

$$y_{p1} = t^2 - 6 t + 14. \tag{33}$$

Note that y_{p1} should be checked when there is time to do so.

- Step 2b. Find y_{p2}.

  As $g_2 = e^{\alpha t} \sin t (a_0 + \cdots + a_n t^n)$ with $\alpha = 0, \beta = 1, n = 0$ we guess

$$y_{p2} = t^s e^{\alpha t} \cos t (A_0 + \cdots + A_n t^n) + t^s e^{\alpha t} \sin t (B_0 + \cdots + B_n t^n) \tag{34}$$

with $\alpha = 0, \beta = 1, n = 0$. Further notice that $\alpha + i\beta = i$ is not a root to the characteristic equation. So $s = 0$.

Therefore

$$y_{p2} = A_0 \cos t + B_0 \sin t. \tag{35}$$

Substitute into the equation:

$$2(A_0 \cos t + B_0 \sin t)'' + 3(A_0 \cos t + B_0 \sin t)' + (A_0 \cos t + B_0 \sin t) = 3 \sin t. \tag{36}$$

This simplifies to

$$(-A_0 + 3 B_0) \cos t + (-B_0 - 3 A_0) \sin t = 3 \sin t = 0 \cdot \cos t + 3 \sin t. \tag{37}$$

Therefore

$$-A_0 + 3 B_0 = 0 \tag{38}$$
$$-B_0 - 3 A_0 = 3. \tag{39}$$

Take the 2nd equation, multiply by 3 and add to the 1st:

$$-10 A_0 = 9 \Longrightarrow A_0 = -\frac{9}{10}. \tag{40}$$

This then gives

$$B = -\frac{3}{10}. \tag{41}$$

So

$$y_{p2} = -\frac{9}{10}\cos t - \frac{3}{10}\sin t. \tag{42}$$

Again y_{p2} should be substituted back into the equation to check whether we have done everything correctly.

- Step 2. Write

$$y_p = y_{p1} + y_{p2} = t^2 - 6\,t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t. \tag{43}$$

- Step 3. The solution to the original problem is

$$y = C_1\,e^{-t/2} + C_2\,e^{-t} + t^2 - 6\,t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t. \tag{44}$$