Lecture 09 2nd Order, Linear, Homogeneous, Constant Coefficient

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Review.

- 2nd Order linear homogeneous equations:

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = 0 \tag{1}$$

or "standard form":

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{2}$$

- General solution:

$$y = C_1\, y_1 + C_2\, y_2. \tag{3}$$

y_1, y_2 form a "fundamental set", that is they are

1. solutions to the equation;
2. linearly independent.

- To check linear independence, use

> Two solutions to the same 2nd order linear equation are linearly independent if and only if their Wronskian $W[y_1, y_2] = y_1'\, y_2 - y_2'\, y_1$ is not zero at some point x_0.

That only one point is enough follows from the following Abel's theorem:

$$W[y_1, y_2](x) = W[y_1, y_2](x_0)\, e^{-\int p} \tag{4}$$

where p is the same $p(x)$ in the standard form of the equation:

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{5}$$

Linear, homogeneous, 2nd order, constant coefficient.

- In other words

$$a\, y'' + b\, y' + c\, y = 0. \tag{6}$$

- How to solve:
  - Step 1: Write down the characteristic equation:

$$a\, r^2 + b\, r + c = 0. \tag{7}$$

  - Step 2: Solve it:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\, a\, c}}{2}. \tag{8}$$

  - Step 3: Write down the general solution.
    - Case 1. r_1, r_2 real and different:[1]

$$y = C_1\, e^{r_1 t} + C_2\, e^{r_2 t}; \tag{9}$$

    - Case 2. r_1, r_2 complex. In this case they have to look like

$$r_1 = \alpha + i\, \beta, \qquad r_2 = \alpha - i\, \beta. \tag{10}$$

The general solution is

$$y = C_1\, e^{\alpha t} \cos \beta t + C_2\, e^{\alpha t} \sin \beta t. \tag{11}$$

1. In most books t instead of x is used when discussing such equations. The reason is that originally most initial value problems for ordinary differential equations come from mechanics where t (time) is the "universal variable". On the other hand, in discussions of boundary value problems x dominates.

- Case 3. $r_1 = r_2$ real. In this case $y_1 = e^{r_1 t}$ and it turns out that $t\,e^{r_1 t}$ always give the 2nd solution (doesn't matter what a, b, c are!). The general solution is

$$y = C_1\,e^{r_1 t} + C_2\,t\,e^{r_1 t}. \tag{12}$$

- Explanations.

  - Why it must be

$$r_1 = \alpha + i\,\beta, \qquad r_2 = \alpha - i\,\beta. \tag{13}$$

Recall the formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}. \tag{14}$$

When r_1 is complex, we necessarily have $b^2 - 4\,a\,c < 0$. So $\sqrt{b^2 - 4\,a\,c} = i\sqrt{4\,a\,c - b^2}$ (this latter square root is a positive number!). Now clearly

$$r_{1,2} = \alpha \pm i\,\beta \tag{15}$$

with

$$\alpha = \frac{-b}{2\,a}, \qquad \beta = \frac{\sqrt{4\,a\,c - b^2}}{2\,a}. \tag{16}$$

We can also reach the same conclusion without using the detailed formula. Recall that if r_1, r_2 solves

$$a\,r^2 + b\,r + c = 0 \tag{17}$$

then the following factorization is true

$$a\,r^2 + b\,r + c = a\,(r - r_1)\,(r - r_2). \tag{18}$$

As

$$(r - r_1)\,(r - r_2) = r^2 - (r_1 + r_2)\,r + r_1\,r_2 \tag{19}$$

we immediately have

$$r_1 + r_2 = -\frac{b}{a} \tag{20}$$

which is real.

  - Why are

$$e^{\alpha t} \cos \beta\,t \text{ and } e^{\alpha t} \sin \beta\,t \tag{21}$$

solutions?

The sleekest way of understanding this is the following. Consider our equation

$$a\,y'' + b\,y' + c\,y = 0 \tag{22}$$

and a complex solution $y_1 = z_1 + i\,z_2$ where z_1, z_2 are real functions. It turns out that z_1, z_2 must both be real solutions of the same equation.

To see this, substitute y be $y_1 = z_1 + i\,z_2$:

$$a\,(z_1 + i\,z_2)'' + b\,(z_1 + i\,z_2)' + c\,(z_1 + i\,z_2) = 0. \tag{23}$$

Expand and organize the left hand side:

$$\begin{aligned} a\,(z_1 + i\,z_2)'' + b\,(z_1 + i\,z_2)' + c\,(z_1 + i\,z_2) &= a\,z_1'' + i\,a\,z_2'' + b\,z_1' + i\,b\,z_2' + c\,z_1 + i\,c\,z_2 \\ &= [a\,z_1'' + b\,z_1' + c\,z_1] + i\,[a\,z_2'' + b\,z_2' + c\,z_2]. \end{aligned} \tag{24}$$

Thus we have

$$[a\,z_1'' + b\,z_1' + c\,z_1] + i\,[a\,z_2'' + b\,z_2' + c\,z_2] = 0. \tag{25}$$

As a complex number being 0 is the same as both its real part and imaginary part are 0, we get

$$a\,z_1'' + b\,z_1' + c\,z_1 = 0; \qquad a\,z_2'' + b\,z_2' + c\,z_2 = 0. \tag{26}$$

- How did we get $t\,e^{r_1 t}$?

  There are many ways. One is called "reduction of order".

  > *Reduction of order*: A method of finding a second solution to a linear differential equation when one solution is already known.

  More specifically, once y_1 is obtained, we try to find a function z such that the product $z\,y_1$ is also a solution. What's beautiful is that, the equation for z is always linear, and further more is always one order less than the equation for y, once we introduce a new unknown $v = z'$. In our case, the equation for $v = z'$ will be 1st order and linear – and can be readily solved.

Example 1. Solve $y'' + 4\,y' + 4\,y = 0$.

First solve the characteristic equation

$$r^2 + 4\,r + 4 = 0 \tag{27}$$

which gives $r_1 = r_2 = -2$. So $y_1 = e^{-2t}$.

To find y_2, set $y_2 = z\,y_1$. Substitute into the equation:

$$(z\,y_1)'' + 4\,(z\,y_1)' + 4\,(z\,y_1) = 0. \tag{28}$$

Compute

$$\begin{aligned}(z\,y_1)' &= z'\,y_1 + z\,y_1'; & (29)\\ (z\,y_1)'' = ((z\,y_1)')' &= (z'\,y_1 + z\,y_1')' = z''\,y_1 + 2\,z'\,y_1' + z\,y_1'' & (30)\end{aligned}$$

Now we have

$$[z''\,y_1 + 2\,z'\,y_1' + z\,y_1''] + 4\,[z'\,y_1 + z\,y_1'] + 4\,z\,y_1 = 0. \tag{31}$$

This can be organized to

$$y_1\,z'' + (2\,y_1' + 4\,y_1)\,z' + [y_1'' + 4\,y_1' + 4\,y_1]\,z = 0. \tag{32}$$

As y_1 is a solution, the last term is 0. The equation for z becomes

$$y_1\,z'' + (2\,y_1' + 4\,y_1)\,z' = 0. \tag{33}$$

Now recall $y_1 = e^{-2t}$. This gives $2\,y_1' + 4\,y_1 = 0$. So finally the equation for z becomes

$$y_1\,z'' = 0 \Longleftrightarrow z'' = 0 \Longleftrightarrow z = C_1 + C_2\,t. \tag{34}$$

Recall that we only need one more solution, we can simply take $z = t$ and get $y_2 = t\,e^{-2t}$. The general solution is then

$$y = C_1\,e^{-2t} + C_2\,t\,e^{-2t}. \tag{35}$$

Remark 2. A different approach is as follows. What we actually get is the following: For any C_1, C_2, $y = z\,y_1 = (C_1 + C_2\,t)\,e^{-2t}$ solves the equation. But this no other than

$$y = C_1\,e^{-2t} + C_2\,t\,e^{-2t} \tag{36}$$

and we have already get the general solution!

Remark 3. When given such a problem in exams, there is no need to "set $y_2 = z\,y_1$" and derive z equation. All you need to do is

1. Solve the characteristic equation;
2. Write down the general solution.

For the above problem, the answer should look like

Solution. The characteristic equation is

$$r^2 + 4\,r + 4 = 0 \tag{37}$$

which has repeated root at $r = -2$. So the general solution is

$$y = C_1 e^{-2t} + C_2 t e^{-2t}. \tag{38}$$

- Examples.
  - Solve the initial value problem

$$y'' + 5y' + 6 = 0; \qquad y(3) = 2;\ y'(3) = 3. \tag{39}$$

Solution. First we get the general solution. Solving the characteristic equation

$$r^2 + 5r + 6 = 0 \tag{40}$$

we get

$$r_{1,2} = -2, -3. \tag{41}$$

So the general solution is

$$y = C_1 e^{-2t} + C_2 e^{-3t}. \tag{42}$$

Now we use the initial conditions to fix the constants. First preparation:

$$y' = -2 C_1 e^{-2t} - 3 C_2 e^{-3t}. \tag{43}$$

So

$$y(3) = 2 \Longrightarrow C_1 e^{-6} + C_2 e^{-9} = 2; \tag{44}$$

$$y'(3) = 3 \Longrightarrow -2 C_1 e^{-6} - 3 C_2 e^{-9} = 3. \tag{45}$$

Multiply the first equation by 2 and add to the second, we get

$$-C_2 e^{-9} = 7 \Longrightarrow C_2 = -7 e^9. \tag{46}$$

Now $C_1 e^{-6} + C_2 e^{-9} = 2$ becomes

$$C_1 e^{-6} - 7 = 2 \Longrightarrow C_1 = 9 e^6. \tag{47}$$

So the final answer is

$$y = 9 e^6 e^{-2t} - 7 e^9 e^{-3t} = 9 e^{6-2t} - 7 e^{9-3t}. \tag{48}$$

Note that the last step of simplification is not required.

  - Solve

$$y'' + 4y' + 5y = 0, \qquad y(3) = 2,\ y'(3) = 3. \tag{49}$$

Solution. First we get general solution. Solving the characteristic equation

$$r^2 + 4r + 5 = 0 \tag{50}$$

we get

$$r_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2} = -2 \pm i. \tag{51}$$

So general solution is

$$y = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t. \tag{52}$$

To use the initial conditions, first prepare

$$y' = [-2 e^{-2t} \cos t - e^{-2t} \sin t]\, C_1 + [-2 e^{-2t} \sin t + e^{-2t} \cos t]\, C_2. \tag{53}$$

Now

$$y(3) = 2 \Longrightarrow C_1 e^{-6} \cos 3 + C_2 e^{-6} \sin 3 = 2; \tag{54}$$

$$y'(3) = 3 \Longrightarrow [-2 e^{-6} \cos 3 - e^{-6} \sin 3]\, C_1 + [-2 e^{-6} \sin 3 + e^{-6} \cos 3]\, C_2 = 3. \tag{55}$$

Simplify a bit:

$$\begin{aligned} (\cos 3)\, C_1 + (\sin 3)\, C_2 &= 2\, e^6 \\ [-2\cos 3 - \sin 3]\, C_1 + [-2\sin 3 + \cos 3]\, C_2 &= 3\, e^6. \end{aligned} \tag{56}$$

Multiply the first equation by $2 + \frac{\sin 3}{\cos 3}$ and add to the second we get

$$\left[\frac{(\sin 3)^2}{\cos 3} + \cos 3 \right] C_2 = 7\, e^6 + \frac{2\sin 3}{\cos 3}\, e^6. \tag{57}$$

Note that

$$\frac{(\sin 3)^2}{\cos 3} + \cos 3 = \frac{1 - (\cos 3)^2}{\cos 3} + \cos 3 = \frac{1}{\cos 3}. \tag{58}$$

So finally

$$C_2 = 7(\cos 3)\, e^6 + 2\,(\sin 3)\, e^6. \tag{59}$$

Substitute back into

$$(\cos 3)\, C_1 + (\sin 3)\, C_2 = 2\, e^6 \tag{60}$$

we get

$$C_1 = 2\,(\cos 3)\, e^6 - 7\,(\sin 3)\, e^6. \tag{61}$$

Final answer:

$$y = [2\cos 3 - 7\sin 3]\, e^{6-2t} \cos t + [7\cos 3 + 2\sin 3]\, e^{6-2t} \sin t. \tag{62}$$