Lecture 07 Existence and Uniqueness

Sep. 21, 2011

How to check your answer.

- If your solution is explicit: $y = Y(x, C)$[1].
  Write your equation in the form $y' = f(x, y)$ or $M(x, y) + N(x, y)\, y' = 0$. Then substitute $y = Y(x, C)$ into the equation. If the solution is correct, the equation should be reduced to identity.
- If your solution is implicit: $u(x, y) = C$.
  Write your equation in the form $M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0$. Then compute

  $$\mathrm{d}u(x, y) = \frac{\partial u(x, y)}{\partial x}\, \mathrm{d}x + \frac{\partial u(x, y)}{\partial y}\, \mathrm{d}y \tag{1}$$

  If $\mathrm{d}u$ and $M\, \mathrm{d}x + N\, \mathrm{d}y$ differ only by a multiplicative factor, that is if there is $\mu(x, y)$ such that

  $$\mathrm{d}u = \mu(x, y)\, (M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y) \tag{2}$$

  then your answer is correct. Otherwise it is not correct.

Example 1. Check whether $x^2 + y^2 = C$ solves $x\, (x^2 + y^2)\, \mathrm{d}x + y\, (x^2 + y^2)\, \mathrm{d}y = 0$.

We compute

$$\mathrm{d}(x^2 + y^2) = 2\, x\, \mathrm{d}x + 2\, y\, \mathrm{d}y = \frac{2}{x^2 + y^2}\, [x\, (x^2 + y^2)\, \mathrm{d}x + y\, (x^2 + y^2)\, \mathrm{d}y]. \tag{3}$$

Therefore the solution is correct.

Existence and Uniqueness.

- Given a DE,
  - Q1:

    Is the solution unique?
  - Q2:

    Does the solution exist?
- The answer to the above is the following theorem.

  Theorem 2. (Existence and Uniqueness) *If $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are bounded for all (x, y) near (x_0, y_0), then the equation*

  $$y' = f(x, y), \qquad y(x_0) = y_0 \tag{4}$$

  has a unique solution at least for x close to x_0.

- Examples.
  - $y' = y$, $y(0) = 0$. We have $x_0 = 0, y_0 = 0, f(x, y) = y$. Thus $\frac{\partial f}{\partial y} = 1$. We see that it is bounded for any x, y. So in particular, the solution to our initial value problem exists and is unique.
  - $y' = y^{1/2}$, $y(0) = 0$. In this case $\frac{\partial f}{\partial y} = \frac{1}{2}\, y^{-1/2}$ which is not bounded for (x, y) near $(0, 0)$. So we cannot expect both existence and uniqueness for this problem. As $y = 0$ is clearly a solution (that is solutions clearly exist), we expect the solution to be not unique, which is indeed the case.

1. The dependence on the arbitrary constant C may or may not be simply $y = Y(x) + C$. For example, the general solution to a linear equation looks like $y = Y(x) + \frac{C}{\mu(x)}$.

- $y' = y^{1/2}, y(0) = 1$. $\frac{\partial f}{\partial y}$ is still $\frac{1}{2} y^{-1/2}$ but it is bounded close to $(x_0, y_0) = (0, 1)$. As a consequence, we expect the solution to this initial value problem to exist and be unique, at least as long as y stays away from 0 (the only value of y that makes $\frac{\partial f}{\partial y}$ infinity). In fact, taking into account the fact that $y^{1/2} \geqslant 0$ and therefore $y' \geqslant 0$ which means y keeps increasing, we see that $y \geqslant 1$ and the existence and uniqueness can be extended to all x.

- Proof of uniqueness.

  The proof consists of the following key steps.

  - Write the equation into the following equivalent integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(\tau, y(\tau)) \, \mathrm{d}\tau. \tag{5}$$

    We check that

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(\tau, y(\tau)) \, \mathrm{d}\tau = y_0 + 0 = y_0; \tag{6}$$

$$y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(y_0) + \frac{\mathrm{d}}{\mathrm{d}x} \int_{x_0}^{x_0} f(\tau, y(\tau)) \, \mathrm{d}\tau = 0 + f(x, y(x)) = f(x, y(x)). \tag{7}$$

  - We need to prove the following: If

$$y(x) = y_0 + \int_{x_0}^{x} f(\tau, y(\tau)) \, \mathrm{d}\tau. \tag{8}$$

    and

$$z(x) = y_0 + \int_{x_0}^{x} f(\tau, z(\tau)) \, \mathrm{d}\tau. \tag{9}$$

    the necessarily $y = z$ for all x. To do this we take the difference:

$$y(x) - z(x) = \int_{x_0}^{x} [f(\tau, y(\tau)) - f(\tau, z(\tau))] \, \mathrm{d}\tau. \tag{10}$$

  - The key idea is to use the following argument: if a number a satisfies

$$|a| \leqslant r\,|a| \tag{11}$$

    for a factor $r < 1$, then $a = 0$.

  - In our case clearly a should be $y(x) - z(x)$. Thus we need to create $y - z$ in the right hand side. This is done by using the Mean Value Theorem:

$$f(a) - f(b) = f'(\xi)\,(a - b) \tag{12}$$

    for some ξ between a, b.

    This leads to

$$f(\tau, y(\tau)) - f(\tau, z(\tau)) = \frac{\partial f}{\partial y}(\xi)\,(y(\tau) - z(\tau)). \tag{13}$$

    Here ξ is some number between $y(\tau)$ and $z(\tau)$.

  - Our equation now becomes

$$y(x) - z(x) = \int_{x_0}^{x} \frac{\partial f}{\partial y}(\xi)\,(y(\tau) - z(\tau)) \, \mathrm{d}\tau. \tag{14}$$

  - Back to our main idea

$$|a| \leqslant r\,|a|, \qquad r < 1 \tag{15}$$

    means $a = 0$. We observe that this is no longer true if we remove the absolute value:

$$a \leqslant r\,a \text{ together with } r < 1 \text{ do not imply } a = 0. \tag{16}$$

So we have to put absolute value on

$$
\begin{aligned}
|y(x) - z(x)| &= \left| \int_{x_0}^{x} \frac{\partial f}{\partial y}(\xi)\,(y(\tau) - z(\tau))\,\mathrm{d}\tau \right| \\
&\leqslant \int_{x_0}^{x} \left| \frac{\partial f}{\partial y}(\xi)\,(y(\tau) - z(\tau)) \right| \mathrm{d}\tau \\
&= \int_{x_0}^{x} \left| \frac{\partial f}{\partial y}(\xi) \right| |y(\tau) - z(\tau)|\,\mathrm{d}\tau
\end{aligned}
\tag{17}
$$

- We know that $\frac{\partial f}{\partial y}$ is bounded. Let M be a constant such that $\left|\frac{\partial f}{\partial y}\right| \leqslant M$. The above now becomes

$$|y(x) - z(x)| \leqslant M \int_{x_0}^{x} |y(\tau) - z(\tau)|\,\mathrm{d}\tau. \tag{18}$$

- Now this is almost[2]

$$|y(x) - z(x)| \leqslant M\,(x - x_0)\,|y(x) - z(x)|. \tag{19}$$

We see that it becomes the

$$|a| \leqslant r\,|a|, \qquad r < 1 \tag{20}$$

situation for all x so close to x_0 that $|x - x_0|\,M < 1$, or equivalently $|x - x_0| < M^{-1}$.

- We have proved: If $y(x_0) = z(x_0)$ then $y(x) = z(x)$ for all $|x - x_0| < M^{-1}$. Now we can take any point x in this interval as the new x_0 and repeat the above argument, and obtain $y(x) = z(x)$ for all $|x - x_0| < 2\,M^{-1}$. Doing this again gives $y = z$ for $|x - x_0| < 3\,M^{-1}$. As this can be done again and again, we see that $y = z$ for all x.[3]

2. See challenge problems of this week.
3. As long as $\left|\frac{\partial f}{\partial y}\right| \leqslant M$ still holds.