Lecture 05 Integrating Factors

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Review.

- Exact equations:

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1}$$

is "exact" when

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

To solve, find $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N \tag{3}$$

and write down the general solution

$$u(x, y) = C. \tag{4}$$

In case of initial value problem, the initial condition is of the form $y(x_0) = y_0$, and the solution becomes

$$u(x, y) = u(x_0, y_0). \tag{5}$$

- Linear equations:

$$y' + p(x)\,y = g(x). \tag{6}$$

Calculate integrating factor:

$$\mu(x) = e^{\int p}. \tag{7}$$

The equation becomes

$$(\mu\,y)' = \mu\,g. \tag{8}$$

Integrate:

$$\mu(x)\,y = \int \mu(x)\,g(x) + C \Longrightarrow y = \frac{1}{\mu(x)} \int \mu(x)\,g(x) + \frac{C}{\mu(x)}. \tag{9}$$

In case of initial value problem, substitute $y = y_0$, $x = x_0$ into the formula of general solutions to obtain C.

Remark 1.

- If the equation is given as

$$a(x)\,y' + b(x)\,y + c(x) = 0, \tag{10}$$

need to first re-write

$$y' + \frac{b(x)}{a(x)}\,y = -\frac{c(x)}{a(x)} \Longrightarrow p(x) = \frac{b(x)}{a(x)}, \quad g(x) = -\frac{c(x)}{a(x)}. \tag{11}$$

- Common mistake:

$$y' = 3\,x\,y + 6 \Longrightarrow p(x) = 3\,x. \tag{12}$$

Separable equations.

The second class of non-exact equations that can be solved easily is **separable equations**, which looks like

$$y' = p(y)\,g(x). \tag{13}$$

To solve this equation, move all y's to left and all x's to right:

$$y' = p(y)\,g(x) \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = p(y)\,g(x) \Longrightarrow \frac{\mathrm{d}y}{p(y)} = g(x)\,\mathrm{d}x. \tag{14}$$

Note the during the last step (dividing by $p(y)$), all constant solutions $y=y_i$, $i=1,2,3,\ldots$, where y_i are such that $p(y_i)=0$, are lost.[1] So we should add them back.

To summarize: The general solution to a separable equation

$$y'=p(y)\,g(x) \tag{15}$$

is

$$P(y)-G(x)=C \text{ and } y=y_i, \qquad i=1,2,3,\ldots \tag{16}$$

with P,G the primitives of $1/p(y)$ and $g(x)$, and y_i satisfies $p(y_i)=0$.

Example 2. Solve

$$y'=3\,x\,y^2. \tag{17}$$

Solution. Separate the variables:

$$\frac{\mathrm{d}y}{y^2}=3\,x\,\mathrm{d}x \tag{18}$$

Integrate

$$-\frac{1}{y}=\frac{3}{2}\,x^2+C\Longrightarrow y=-\frac{1}{\frac{3}{2}\,x^2+C}. \tag{19}$$

Add back the constant solutions:

$$y^2=0\Longrightarrow y=0. \tag{20}$$

So the answer is

$$y=-\frac{1}{\frac{3}{2}\,x^2+C} \text{ and } y=0. \tag{21}$$

Homogeneous equation.

A “homogeneous equation” is of the form

$$y'=H(y/x). \tag{22}$$

It can be transformed to separable through setting $v=y/x$.

Example 3. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2+y^2}{2\,x\,y}. \tag{23}$$

Solution.

Notice that the right hand side is simply $\frac{1}{2}\frac{x}{y}+\frac{1}{2}\frac{y}{x}$ so the equation is homogeneous. Let $v=y/x$. We have $y=v\,x\Longrightarrow y'=x\,v'+v$. So the equation becomes

$$x\,v'+v=\frac{1}{2}\frac{1}{v}+\frac{1}{2}\,v\Longrightarrow x\,v'=\frac{1-v^2}{2\,v}. \tag{24}$$

Separate the variables:

$$\frac{2\,v\mathrm{d}v}{1-v^2}=\frac{\mathrm{d}x}{x}. \tag{25}$$

Integrate

$$\int\frac{2\,v\,\mathrm{d}v}{1-v^2}=-\int\frac{2\,v\,\mathrm{d}v}{v^2-1}=-\int\frac{\mathrm{d}v^2}{v^2-1}=-\int\frac{\mathrm{d}(v^2-1)}{v^2-1}=-\ln|v^2-1|, \qquad \int\frac{\mathrm{d}x}{x}=\ln|x|. \tag{26}$$

So we get

$$\ln|(v^2-1)\,x|=C. \tag{27}$$

Add back the constant solutions:

$$\frac{1-v^2}{2\,v}=0\Longrightarrow v=\pm 1. \tag{28}$$

So the solutions (for the v equation) are

$$\ln|(v^2-1)\,x|=C, \qquad v=\pm 1. \tag{29}$$

1. Those who are curious can try to show that these are indeed all the constant solutions to the equation.

We can simplify this through the following.

$$\begin{aligned} & \ln|(v^2-1)\,x| = \text{arbitrary constant} \\ \text{is the same as}\quad & |(v^2-1)\,x| = e^{\text{arbitrary constant}} \\ \text{is the same as}\quad & |(v^2-1)\,x| = \text{arbitrary positive constant} \\ \text{is the same as}\quad & (v^2-1)\,x = \text{arbitrary non-zero constant.} \end{aligned}$$

Now it is clear that

$$\ln|(v^2-1)\,x| = C, \qquad v = \pm 1. \tag{30}$$

is the same as

$$(v^2-1)\,x = C \tag{31}$$

where C is an arbitrary constant.

Finally we go back to y. Recall $v = y/x$. So the solution becomes

$$\left(\frac{y^2}{x^2} - 1\right) x = C \tag{32}$$

which simplify to

$$y^2 - x^2 = C\,x. \tag{33}$$

General non-exact equations.

- Let's review what exactly has been done.
  - For exact equation, we solve it.
  - For linear equation, we multiply by $\mu(x) = e^{\int p}$. Now take a closer look:

    $$y' + p(x)\,y = g(x) \Longleftrightarrow [p(x)\,y - g(x)]\,\mathrm{d}x + \mathrm{d}y = 0 \tag{34}$$

    Multiply by $\mu(x)$:

    $$[\mu(x)\,p(x)\,y - \mu(x)\,g(x)]\,\mathrm{d}x + \mu(x)\,\mathrm{d}y = 0. \tag{35}$$

    Check

    $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[\mu(x)\,p(x)\,y - \mu(x)\,g(x)] = \mu(x)\,p(x), \tag{36}$$

    $$\frac{\partial N}{\partial x} = \frac{\partial \mu(x)}{\partial x} = \mu'(x). \tag{37}$$

    They are the same exactly when $\mu = e^{\int p}$!

    By multiplying the equation using μ, we have transformed the equation to exact.
  - For separable equation, we divide by $\frac{1}{p(y)}$. The equation changes to

    $$-g(x)\,\mathrm{d}x + \frac{1}{p(y)}\,\mathrm{d}y = 0 \tag{38}$$

    which is clearly exact.
- Summary:

  The only strategy we have used so far is: Multiply the equation by some "integrating factor" and make the equation exact.
  - For linear equation, the integrating factor is a function of x;
  - For separable equation, the integrating factor is a function of y.
- What if we use a general function $\mu(x, y)$? Will we be able to transform any equation to an exact one?
  - In theory, yes. For any equation

    $$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{39}$$

there exists $\mu(x, y)$ such that

$$(\mu(x, y)\, M(x, y))\, \mathrm{d}x + (\mu(x, y)\, N(x, y))\, \mathrm{d}y = 0 \tag{40}$$

is exact.

- In practice, no.

  Reason: To find such $\mu(x, y)$ in the general case, we need to solve the so-called "characteristics equation" first. This "characteristics equation" is exactly

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{41}$$

- The equation satisfied by μ:

$$\frac{\partial}{\partial y}(\mu(x, y)\, M(x, y)) = \frac{\partial}{\partial x}(\mu(x, y)\, N(x, y)) \tag{42}$$

  simplifies to the equation for μ:

$$M(x, y)\, \frac{\partial \mu}{\partial y} - N(x, y)\, \frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mu. \tag{43}$$

- A suggetion: Instead of remembering the above equation, it is a better idea (less prone to mistakes) to remember how it is derived: $\mu\, M\, \mathrm{d}x + \mu\, N\, \mathrm{d}y = 0$ is exact.