Lecture 04 Simplest Non-Exact Equations

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Review.

- An equation
$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1}$$
(Can also be written as
$$y' = f(x,y) \text{ or } M(x,y) + N(x,y)\,y' = 0 \tag{2}$$
The relation to the former can we seen from setting $f(x,y) = -\frac{M}{N}$) is said to be **exact** if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{3}$$

- An exact equation can be solved as follows
  1. Find $u(x,y)$ such that $\frac{\partial u}{\partial x} = M; \frac{\partial u}{\partial y} = N$ through either one of the following approaches:
     - Approach #1:
       a. Evaluate
       $$u(x,y) = \int M\,\mathrm{d}x + g(y); \tag{4}$$
       b. Determine $g(y)$ using
       $$\frac{\partial u}{\partial y} = N(x,y) \tag{5}$$
     - Approach #2:
       a. Evaluate
       $$u(x,y) = \int N\,\mathrm{d}y + g(x) \tag{6}$$
       b. Determine $g(x)$ using
       $$\frac{\partial u}{\partial x} = M(x,y). \tag{7}$$
  2. Write down the general solution
  $$u(x,y) = C. \tag{8}$$
  3. (If an initial value problem). If the initial condition reads $y(x_0) = y_0$, then necessarily
  $$u(x_0, y_0) = C \tag{9}$$
  and the solution for the initial value problem is
  $$u(x,y) = u(x_0, y_0). \tag{10}$$

- An example (§2.6 9)
$$(y\,e^{xy}\cos 2\,x - 2\,e^{xy}\sin 2\,x + 2\,x)\,\mathrm{d}x + (x\,e^{xy}\cos 2\,x - 3)\,\mathrm{d}y = 0, \qquad y(3) = 7. \tag{11}$$
Solution. This is an initial value problem. We solve it through the above three steps.
  1. Find u. Since
  $$M = y\,e^{xy}\cos 2\,x - 2\,e^{xy}\sin 2\,x + 2\,x, \qquad N = x\,e^{xy}\cos 2\,x - 3 \tag{12}$$
  we have two choices:
  $$u = \int (y\,e^{xy}\cos 2\,x - 2\,e^{xy}\sin 2\,x + 2\,x)\,\mathrm{d}x + g(y) \text{ or } u = \int (x\,e^{xy}\cos 2\,x - 3)\,\mathrm{d}y + g(x) \tag{13}$$

The decision should be made basing on the relative difficulty of the two integrals. To make things clear, we grey out the variable that will be treated as constant when integrating:

$$\int (y\, e^{x y} \cos 2\, x - 2\, e^{x y} \sin 2\, x + 2\, x)\, \mathrm{d}x \text{ or } \int (x\, e^{x y} \cos 2\, x - 3)\, \mathrm{d}y \tag{14}$$

Now it's very clear that the second integral would be much simpler to do.

We calculate

$$\begin{aligned}\int (x\, e^{x y} \cos 2\, x - 3)\, \mathrm{d}y &= x \cos 2\, x \int e^{x y}\, \mathrm{d}y - 3 \int \mathrm{d}y \\ &= x \cos 2\, x \left(\frac{e^{x y}}{x} \right) - 3\, y \\ &= e^{x y} \cos 2\, x - 3\, y.\end{aligned} \tag{15}$$

Thus all we need to do is to find $g(x)$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (e^{x y} \cos 2\, x - 3\, y + g(x)) = M(x, y) = y\, e^{x y} \cos 2\, x - 2\, e^{x y} \sin 2\, x + 2\, x \tag{16}$$

Calculating the partial derivative

$$\begin{aligned}\frac{\partial}{\partial x} (e^{x y} \cos 2\, x - 3\, y + g(x)) &= \frac{\partial}{\partial x} (e^{x y} \cos 2\, x) - \frac{\partial}{\partial x} (3\, y) + \frac{\partial}{\partial x} g(x) \\ &= \frac{\partial}{\partial x} (e^{x y}) \cos 2\, x + e^{x y} \frac{\partial}{\partial x} (\cos 2\, x) - 0 + g'(x) \\ &= y\, e^{x y} \cos 2\, x - 2\, e^{x y} \sin 2\, x + g'(x)\end{aligned} \tag{17}$$

and compare with $M(x, y)$, we see that $g'(x) = 2\, x \Longrightarrow g(x) = x^2$.[1]

So

$$u(x, y) = e^{x y} \cos 2\, x - 3\, y + x^2. \tag{18}$$

2. The general solution is

$$e^{x y} \cos 2\, x - 3\, y + x^2 = C. \tag{19}$$

3. $y(3) = 7$ so

$$C = e^{3 \times 7} \cos (2 \times 3) - 3 \times 7 + 3^2 = e^{21} \cos 6 - 12. \tag{20}$$

Finally the solution to the initial value problem is

$$e^{x y} \cos 2\, x - 3\, y + x^2 = e^{21} \cos 6 - 12. \tag{21}$$

Linear equations.

Fortunately there are many important equations that are exact, unfortunately there are many more that are not.

- The simplest non-exact equation.

$$y' = y \text{ or equivalently } -y\, \mathrm{d}x + \mathrm{d}y = 0. \tag{22}$$

  We easily check

$$\frac{\partial M}{\partial y} = -1 \neq 0 = \frac{\partial N}{\partial x}. \tag{23}$$

- But it can be easily solved!
  - Approach 1: Remember that $(e^x)' = e^x$ itself. Then realize after a while that this is also true for $C\, e^x$ for any constant C. As we have one arbitrary constant now, the general solution is

$$y = C\, e^x. \tag{24}$$

  - Approach 2: Write the equation as

$$y' - y = 0. \tag{25}$$

1. We only need one primitive here. So instead of writing $g(x) = x^2 + C$ we just use $g(x) = x^2$.

Then recall that

$$(e^{-x} y)' = (e^{-x})' y + e^{-x} y' = -e^{-x} y + e^{-x} y' = e^{-x} (y' - y). \tag{26}$$

Therefore multiplying the equation by e^{-x} makes the left hand side a total derivative:

$$(e^{-x} y)' = e^{-x} (y' - y) = 0 \Longrightarrow e^{-x} y = C \Longrightarrow y = C e^{x}. \tag{27}$$

It is clear that the 2nd approach is much more systematic.

- Indeed it can be generalized to solve all equations of the form (called "***linear equations***").

$$y' + p(x) y = g(x). \tag{28}$$

  - Step 0: If the equation is given as

    $$a(x) y' + b(x) y = c(x), \tag{29}$$

    divide by $a(x)$ to reach the above "standard" form (that is $p(x) = b(x)/a(x)$, $g(x) = c(x)/a(x)$).

  - Step 1: Compute the "integrating factor":

    $$\mu(x) = e^{\int p(x) \mathrm{d}x}. \tag{30}$$

  - Step 2: Multiplying the equation (**the one in standard form!**) by $\mu(x)$ to get

    $$(\mu(x) y)' = \mu(x) y' + \mu(x) p(x) y = \mu(x) g(x). \tag{31}$$

    Note. It's a good idea to check the red "=" to make sure that they really equal! In other words, we should check whether our calculation of μ is correct or not!

  - Step 3: Integrate:

    $$\mu(x) y = \int \mu(x) g(x) + C. \tag{32}$$

  - Step 4: Divide:

    $$y = \frac{1}{\mu(x)} \int \mu(x) g(x) + \frac{C}{\mu(x)}. \tag{33}$$

  - Step 5: If there is an initial condition $y(x_0) = y_0$, substitute into the solution

    $$y_0 = \frac{1}{\mu(x_0)} \int^{x_0} \mu(x) g(x) + \frac{C}{\mu(x_0)} \tag{34}$$

    to determine C.

- Example. Solve

$$x^2 y' + 4 x y = e^x. \tag{35}$$

Solution. First write it into "standard form":

$$y' + \frac{4}{x} y = \frac{e^x}{x^2}. \tag{36}$$

We see that $p(x) = \frac{4}{x}$, $g(x) = \frac{e^x}{x^2}$.

Now compute that integrating factor:

$$\mu(x) = e^{\int p} = \exp\left[\int \frac{4}{x} \mathrm{d}x\right] = \exp[4 \ln |x|] = \exp[\ln x^4] = x^4. \tag{37}$$

Multiply the standard form equation by x^4 we get

$$x^4 y' + 4 x^3 y = x^2 e^x \tag{38}$$

which should be just

$$(x^4 y)' = x^2 e^x. \tag{39}$$

We check that indeed

$$(x^4\, y)' = x^4\, y' + 4\, x^3\, y \tag{40}$$

So our calculation of the integrating factor was correct.

We have

$$\begin{aligned}
x^4\, y &= \int x^2\, e^x\, \mathrm{d}x + C \\
& \quad (u = x^2, v = e^x) \\
&= \int u\, \mathrm{d}v + C \\
&= u\, v - \int v\, \mathrm{d}u + C \\
&= x^2\, e^x - \int e^x\, \mathrm{d}(x^2) + C \\
&= x^2\, e^x - 2 \int x\, e^x\, \mathrm{d}x + C \\
&= x^2\, e^x - 2 \int x\, \mathrm{d}e^x + C \qquad \text{(We omitted the setting of } u = x, v = e^x\text{)} \\
&= x^2\, e^x - 2\, x\, e^x + 2 \int e^x\, \mathrm{d}x + C \\
&= (x^2 - 2\, x + 2)\, e^x + C.
\end{aligned} \tag{41}$$

So finally the general solution is

$$y = \frac{x^2 - 2\, x + 2}{x^4}\, e^x + \frac{C}{x^4}. \tag{42}$$

The road not taken.

Back to the example

$$(y\, e^{x y} \cos 2\, x - 2\, e^{x y} \sin 2\, x + 2\, x)\, \mathrm{d}x + (x\, e^{x y} \cos 2\, x - 3)\, \mathrm{d}y = 0, \qquad y(3) = 7. \tag{43}$$

After looking down as far as we could, we choose to do write $u = \int N\, \mathrm{d}y + g(x)$. What if we choose the other one? Let's see.

Note. Anyone who's not curious about this should stop here. The following has no direct relation to the exams.

We first integrate $\int M\, \mathrm{d}x$. We have

$$\int (y\, e^{x y} \cos 2\, x - 2\, e^{x y} \sin 2\, x + 2\, x)\, \mathrm{d}x = \int y\, e^{x y} \cos 2\, x\, \mathrm{d}x - 2 \int e^{x y} \sin 2\, x + 2 \int x\, \mathrm{d}x. \tag{44}$$

To compute the first term we need to evaluate

$$\begin{aligned}
\int e^{x y} \cos 2\, x\, \mathrm{d}x &= \frac{1}{2} \int e^{x y}\, \mathrm{d}\sin 2\, x \\
&= \frac{1}{2} \left[e^{x y} \sin 2\, x - \int \sin 2\, x\, \mathrm{d}e^{x y} \right] \\
&= \frac{1}{2} \left[e^{x y} \sin 2\, x - y \int e^{x y} \sin 2\, x\, \mathrm{d}x \right] \\
&= \frac{1}{2}\, e^{x y} \sin 2\, x - \frac{y}{2} \int e^{x y} \sin 2\, x\, \mathrm{d}x \\
&= \frac{1}{2}\, e^{x y} \sin 2\, x + \frac{y}{4} \int e^{x y}\, \mathrm{d}\cos 2\, x \\
&= \frac{1}{2}\, e^{x y} \sin 2\, x + \frac{y}{4} \left[e^{x y} \cos 2\, x - \int \cos 2\, x\, \mathrm{d}e^{x y} \right] \\
&= \frac{1}{2}\, e^{x y} \sin 2\, x + \frac{y\, e^{x y} \cos 2\, x}{4} - \frac{y^2}{4} \int e^{x y} \cos 2\, x\, \mathrm{d}x.
\end{aligned} \tag{45}$$

Highlighting the first and the last:

$$\int e^{xy}\cos 2x\,\mathrm{d}x = \frac{1}{2}e^{xy}\sin 2x + \frac{y\,e^{xy}\cos 2x}{4} - \frac{y^2}{4}\int e^{xy}\cos 2x\,\mathrm{d}x \tag{46}$$

we get

$$\left(1+\frac{y^2}{4}\right)\int e^{xy}\cos 2x\,\mathrm{d}x = \frac{1}{2}e^{xy}\sin 2x + \frac{y\,e^{xy}\cos 2x}{4} \tag{47}$$

which gives

$$\int e^{xy}\cos 2x\,\mathrm{d}x = \frac{2\,e^{xy}\sin 2x + y\,e^{xy}\cos 2x}{(4+y^2)}. \tag{48}$$

Therefore the first integral is

$$\int y\,e^{xy}\cos 2x\,\mathrm{d}x = \frac{2\,y\,e^{xy}\sin 2x + y^2\,e^{xy}\cos 2x}{(4+y^2)}. \tag{49}$$

The second integral can be evaluated similarly, but we take a short cut by noticing that $\int\, e^{xy}\sin 2\,x\,\mathrm{d}x$ already appears in line 3 of the above calculation of the first integral. So we have

$$\int e^{xy}\cos 2x\,\mathrm{d}x = \frac{1}{2}\left[e^{xy}\sin 2x - y\int e^{xy}\sin 2x\,\mathrm{d}x\right] \tag{50}$$

which gives

$$\int e^{xy}\sin 2x\,\mathrm{d}x = \frac{1}{y}e^{xy}\sin 2x - \frac{4\,e^{xy}\sin 2x + 2\,y\,e^{xy}\cos 2x}{y\,(4+y^2)} = \frac{y\,e^{xy}\sin 2x - 2\,e^{xy}\cos 2x}{4+y^2}. \tag{51}$$

So finally we can write

$$\begin{aligned}\int (y\,e^{xy}\cos 2x - 2\,e^{xy}\sin 2x + 2x)\,\mathrm{d}x &= \int y\,e^{xy}\cos 2x\,\mathrm{d}x - 2\int e^{xy}\sin 2x + 2\int x\,\mathrm{d}x\\ &= \frac{2\,y\,e^{xy}\sin 2x + y^2\,e^{xy}\cos 2x}{(4+y^2)} - \frac{2\,y\,e^{xy}\sin 2x - 4\,e^{xy}\cos 2x}{4+y^2} + x^2\\ &= e^{xy}\cos 2x + x^2.\end{aligned}$$

Therefore

$$u(x,y) = e^{xy}\cos 2x + x^2 + g(y). \tag{52}$$

To determine $g(y)$ we calculate

$$\frac{\partial u}{\partial y} = x\,e^{xy}\cos 2x + g'(y) \tag{53}$$

and compare with

$$N(x,y) = x\,e^{xy}\cos 2x - 3 \tag{54}$$

to obtaine

$$g'(y) = -3 \Longrightarrow g(y) = -3\,y. \tag{55}$$

So

$$u(x,y) = e^{xy}\cos 2x + x^2 - 3\,y. \tag{56}$$

We have obtained the same result!