Lecture 03 Exact Equations

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Review.

	Differential Equation	General Solution
Simplest	$y' = f(x)$	$y = F(x) + C$ here $F' = f$
Generalization	$\mathrm{d}u(x, y) = 0$	$u(x, y) = C$

Table 1. Equations we can solve so far

- Equations we will meet:
  - $y' = f(x, y)$;
  - $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$;
  - $M(x, y) + N(x, y)\,y' = 0$.

  Note: These are just different ways of writing the same equation.
- How to "bridge the gap"?
  - Try to establish relation between $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ and $\mathrm{d}u = 0$.

Exact equations.

- Recall

$$\mathrm{d}u(x, y) = \frac{\partial u(x, y)}{\partial x}\,\mathrm{d}x + \frac{\partial u(x, y)}{\partial y}\,\mathrm{d}y. \tag{1}$$

Example 1. Compute $\mathrm{d}(e^{x y} + x^3\,y)$.

We compute

$$\begin{aligned} \frac{\partial}{\partial x}(e^{x y} + x^3\,y) &= \frac{\partial(e^{x y})}{\partial x} + \frac{\partial(x^3\,y)}{\partial x} \\ &= e^{x y}\,\frac{\partial(x\,y)}{\partial x} + 3\,x^2\,y \quad \left(\text{Note that } y \text{ is just "constant" to } \frac{\partial}{\partial x}!\right) \\ &= y\,e^{x y} + 3\,x^2\,y. \end{aligned} \tag{2}$$

Similarly we can compute

$$\frac{\partial}{\partial y}(e^{x y} + x^3\,y) = x\,e^{x y} + x^3 \tag{3}$$

Therefore

$$\mathrm{d}(e^{x y} + x^3\,y) = (y\,e^{x y} + 3\,x^2\,y)\,\mathrm{d}x + (x\,e^{x y} + x^3)\,\mathrm{d}y. \tag{4}$$

- Give $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$, if we can find $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N \tag{5}$$

then

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \text{ becomes } \mathrm{d}u(x, y) = 0 \tag{6}$$

and the general solution is

$$u(x, y) = C. \tag{7}$$

Example 2. We already know

$$\mathrm{d}(e^{x y} + x^3\,y) = (y\,e^{x y} + 3\,x^2\,y)\,\mathrm{d}x + (x\,e^{x y} + x^3)\,\mathrm{d}y \tag{8}$$

so we can immediately get the general solution of

$$(y\,e^{x\,y}+3\,x^2\,y)\,\mathrm{d}x+(x\,e^{x\,y}+x^3)\,\mathrm{d}y=0 \tag{9}$$

as

$$e^{x\,y}+x^3\,y=C. \tag{10}$$

Note that this is an "implicit formula" for the solution. And it is impossible to write the above as $y=\cdots$.[1]

- An equation $M\,\mathrm{d}x+N\,\mathrm{d}y=0$ such that there is u satisfying $\mathrm{d}u=M\,\mathrm{d}x+N\,\mathrm{d}y$ is called "**exact**".
- Two questions:
  - Q1: Are all (1st order) differential equations exact?
  - Q2: When an equation is exact, how do we find u?
- How do we find u? (Ans to Q2)
  - Bruteforce integration.

Example 3. We know that $(y\,e^{x\,y}+3\,x^2\,y)\,\mathrm{d}x+(x\,e^{x\,y}+x^3)\,\mathrm{d}y=0$ is exact. Find u.

We need u to satisfy

$$\frac{\partial u}{\partial x}=y\,e^{x\,y}+3\,x^2\,y; \qquad \frac{\partial u}{\partial y}=x\,e^{x\,y}+x^3. \tag{11}$$

The idea is to use the two conditions one by one.

– We can first use

$$\frac{\partial u}{\partial x}=y\,e^{x\,y}+3\,x^2\,y. \tag{12}$$

Integrate in x: (when doing this, remember that y can be treated as constant!)

$$\begin{aligned} u(x,y) &= \int (y\,e^{x\,y}+3\,x^2\,y)\,\mathrm{d}x \\ &= \int y\,e^{x\,y}\,\mathrm{d}x+\int 3\,x^2\,y\,\mathrm{d}x \\ &= \int e^{x\,y}\,\mathrm{d}(x\,y)+y\int 3\,x^2\,\mathrm{d}x \\ &= e^{x\,y}+x^3\,y+g(y). \end{aligned} \tag{13}$$

Notice that instead of "arbitrary constant" we have "arbitrary function of y". This is because when integrating with respect to x, y should be treated as constant.

To summarize,

$$\frac{\partial u}{\partial x}=y\,e^{x\,y}+3\,x^2\,y \tag{14}$$

requires our u to take the special form

$$u(x,y)=e^{x\,y}+x^3\,y+g(y). \tag{15}$$

– Now use

$$\frac{\partial u}{\partial y}=x\,e^{x\,y}+x^3 \tag{16}$$

Our partial knowledge of u gives

$$x\,e^{x\,y}+x^3=\frac{\partial}{\partial y}[e^{x\,y}+x^3\,y+g(y)]=x\,e^{x\,y}+x^3+g'(y) \tag{17}$$

1. The reason is that $e^x=x+C$ is a "transcendental equation" whose solution cannot be written down as formulas involving our familiar functions.

which reduces to

$$g'(y) = 0. \tag{18}$$

In other words, if we take any $g(y)$ satisfying $g'(y) = 0$ and write $u(x, y) = e^{xy} + x^3 y + g(y)$, we would have a u satisfying

$$\frac{\partial u}{\partial x} = y e^{xy} + 3 x^2 y; \qquad \frac{\partial u}{\partial y} = x e^{xy} + x^3. \tag{19}$$

– So

$$u(x, y) = e^{xy} + x^3 y + C \tag{20}$$

for any constant C.

Remark 4. We can also start from $\frac{\partial u}{\partial y} = x e^{xy} + x^3$ and obtain the same result.

◦ Algorithm.

Given $M(x, y), N(x, y)$, find $u(x, y)$ such that $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$.

– Approach 1.

1. Step 1. Write

$$u(x, y) = \int M(x, y)\, \mathrm{d}x + g(y). \tag{21}$$

2. Step 2. Determine g through

$$N(x, y) = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x, y)\, \mathrm{d}x \right) + g'(y). \tag{22}$$

3. Step 3. Write down $u(x, y)$.

– Approach 2.

1. Step 1. Write

$$u(x, y) = \int N(x, y)\, \mathrm{d}y + f(x). \tag{23}$$

2. Step 2. Determine f through

$$M(x, y) = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\int N(x, y)\, \mathrm{d}y \right) + f'(x). \tag{24}$$

3. Step 3. Write down $u(x, y)$.

Which approach to take: If $\int M\, \mathrm{d}x$ looks harder to do than $\int N\, \mathrm{d}y$, take Approach 1; If $\int N\, \mathrm{d}y$ seems harder, take Approach 2.

• Are all 1st order DEs exact? (Ans to Q1)

◦ Ans: NO.

◦ How do we know an equation is exact?

– Notice: If $M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$, then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}. \tag{25}$$

In other words, if $M\, \mathrm{d}x + N\, \mathrm{d}y = 0$ is exact, then necessarily

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{26}$$

– Checking exactness.

Theorem 5. *Given an equation $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the equation is exact, and the solution is given by $u(x, y) = C$ where u satisfies*

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N. \tag{27}$$

Remark 6. Note that if u is OK, $u + C$ is also OK for any constant C. However this does not change the general solution we obtain.

Proof. (by construction). We write

$$u(x, y) = \int M(x, y)\,\mathrm{d}x + g(y) \tag{28}$$

and show that there exists a $g(y)$ such that

$$\frac{\partial u}{\partial x} = M; \quad \frac{\partial u}{\partial y} = N. \tag{29}$$

As

$$\frac{\partial}{\partial x}\left(\int M\,\mathrm{d}x + g(y)\right) = \frac{\partial}{\partial x}\int M\,\mathrm{d}x + \frac{\partial g(y)}{\partial x} = M(x, y) + 0 = M \tag{30}$$

all we need to do is to check the 2nd condition.

Compute

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M\,\mathrm{d}x + g(y)\right) = \int \frac{\partial M}{\partial y}\,\mathrm{d}x + g'(y). \tag{31}$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, we further have

$$\frac{\partial u}{\partial y} = \int \frac{\partial N}{\partial x}\,\mathrm{d}x + g'(y) \tag{32}$$

Thus the requirement on g is

$$g'(y) = N - \int \frac{\partial N}{\partial x}\,\mathrm{d}x. \tag{33}$$

Now notice that such g exists when the right hand side is independent of x. We check

$$\frac{\partial}{\partial x}\left[N - \int \frac{\partial N}{\partial x}\,\mathrm{d}x\right] = \frac{\partial N}{\partial x} - \frac{\partial N}{\partial x} = 0. \tag{34}$$

Indeed! □

- Solving exact equations.

Example 7. Solve

$$(4\,x^3 + 3\,y)\,\mathrm{d}x + (e^y + 3\,x)\,\mathrm{d}y = 0. \tag{35}$$

Solution. We have $M = (4\,x^3 + 3\,y)$ and $N = (e^y + 3\,x)$.

First check exactness:

$$\frac{\partial M}{\partial y} = 3; \qquad \frac{\partial N}{\partial x} = 3. \tag{36}$$

The equation is exact![2]

Now we find u. It seems both

$$\int (4\,x^3 + 3\,y)\,\mathrm{d}x \text{ and } \int (e^y + 3\,x)\,\mathrm{d}y \tag{37}$$

2. Trying to find u without checking exactness is a bad idea. Much time will have been wasted before realizing that no such u exists.

are easy. So we just choose to start from

$$u(x,y) = \int (4x^3 + 3y)\,\mathrm{d}x + g(y) = x^4 + 3xy + g(y). \tag{38}$$

Now

$$e^y + 3x = N = \frac{\partial}{\partial y}(x^4 + 3xy + g(y)) = 3x + g'(y). \tag{39}$$

We see that the requirement for g is

$$g'(y) = e^y. \tag{40}$$

The solution is

$$g(y) = e^y + C \tag{41}$$

but as we are free to choose the value of C, we just choose $C = 0$.

Putting things together we have

$$u(x,y) = x^4 + 3xy + e^y \tag{42}$$

and the general solution to the original problem is

$$x^4 + 3xy + e^y = C. \tag{43}$$

When time allows, we should check:

$$\begin{aligned} 0 = \mathrm{d}(x^4 + 3xy + e^y) &= \frac{\partial}{\partial x}(x^4 + 3xy + e^y)\,\mathrm{d}x + \frac{\partial}{\partial y}(x^4 + 3xy + e^y)\,\mathrm{d}y \\ &= (4x^3 + 3y)\,\mathrm{d}x + (e^y + 3x)\,\mathrm{d}y \\ &\quad \text{(Compare with the equation)} \\ &= 0. \end{aligned} \tag{44}$$