Oct. 28, 2011

MATH 334 FALL 2011 HOMEWORK 8

BASIC

Problem 1. Find the general solution for the following:

a) $x^2\, y'' + 4\, x\, y' + 2\, y = 0$.

b) $x^2\, y'' + 5\, x\, y' + 4\, y = 0$.

c) $2\, x^2\, y'' + 3\, x\, y' + 4\, y = 0$.

Problem 2. Find all singular points of

$$x^2\, (1 - x)\, y'' + (x - 2)\, y' - 3\, x\, y = 0, \tag{1}$$

and determine whether each one is regular or irregular.

INTERMEDIATE

Problem 3. Determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the differential equation

$$(1 + x^3)\, y'' + 4\, x\, y' + 4\, y = 0; \qquad x_0 = 0,\ x_0 = 2. \tag{2}$$

ADVANCED

Problem 4. Find the first five terms of the power series solution for

$$x\, y'' + y \ln(1-x) = 0 \tag{3}$$

and determine a lower bound for its radius of convergence.

Problem 5. Consider the equation

$$2\, x^2\, y'' + x\,(2\,x+1)\, y' - y = 0. \tag{4}$$

a) Is 0 a regular(ordinary) point, a regular singular point, or an irregular singular point?

b) Write down and solve the indicial equation.

c) Write down the correct forms of y_1, y_2.

d) If the two roots of the indicial equation does not differ by an integer, find y_1, y_2.

CHALLENGE

Problem 6. Construct an example of an equation that does not have a solution of the form $x^\alpha \sum_{n=0}^\infty a_n\, x^n$. (Hint: What equation does e^{1/x^2} solve?)

Problem 7. Prove the following: If $f(x) = \sum_{n=0}^\infty a_n\,(x-x_0)^n$ for $|x-x_0| < R$ for some $R > 0$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{5}$$

In other words, if f is analytic at some point x_0, then the corresponding power series is necessarily the Taylor expansion of f.

Problem 8. Show that Euler equations

$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0 \tag{6}$$

can be transformed to 2nd order constant-coefficient linear equations through the change of variable: $t = \ln x$. Write down that equation.

Problem 9. Find a function $p(x)$ for which $\lim_{x\to x_0} (x-x_0)\, p(x)$ is finite, but $(x-x_0)\, p(x)$ is **not** analytic at x_0. Then prove that if $p(x)$ is rational, that is $p(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials, then the finiteness of the above limit indeed implies the analyticity of $(x-x_0)\, p$.

Problem 10. Prove the following. If all solutions to $y'' + p(x)\, y' + q(x) y = 0$ are analytic at $x_0 = 0$, then p, q are analytic there too.

See Next Page for Answers

Answers

- Problem 1

  a) $y = C_1 x^{-2} + C_2\, x^{-1}$.

  b) $y = C_1\, x^{-2} + C_2\, x^{-2} \ln x$.

  c) $y = C_1\, x^{-1/4} \cos\left(\frac{\sqrt{31}}{4} \ln x\right) + C_2\, x^{-1/4} \sin\left(\frac{\sqrt{31}}{4} \ln x\right)$.

- Problem 2

  0 (irregular), 1 (regular)

- Problem 3

  $x_0 = 0$: 1; $x_0 = 2$: $\sqrt{3}$.

- Problem 4

  $y = a_0 + a_1\, x + \frac{a_0}{2}\, x^2 + \left(\frac{a_0}{12} + \frac{a_1}{6}\right) x^3 + \left(\frac{a_0}{72} + \frac{a_1}{24}\right) x^4 + \cdots$

- Problem 5

  a) Regular singular;

  b) $1, -\frac{1}{2}$;

  c) $y_1 = x \sum_{n=0}^{\infty} a_n\, x^n, \qquad y_2 = x^{-1/2} \sum_{n=0}^{\infty} b_n\, x^n$.

  d) $y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right) \cdots \frac{5}{2}}\, x^{n+1}$. $y_2 = x^{-\frac{1}{2}}\, e^{-x}$.

- Problem 8

  $a\, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (b - a)\, \frac{\mathrm{d}y}{\mathrm{d}t} + c\, y = 0$.

- Problem 9

  $\frac{1}{x}\, e^{-\frac{1}{x^2}}$.