Oct. 28, 2011

Math 334 Fall 2011 Homework 7 Solutions

Basic

Problem 1. Determine the radius of convergence for the following:

a) $\sum_{n=0}^{\infty} n^2 (x-1)^n$.

b) $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}$.

c) $\sum_{n=0}^{\infty} \frac{n!\,x^n}{n^n}$.

Solution.

a) We have $a_n = n^2$; So

$$\rho^{-1} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1. \tag{1}$$

Therefore the radius of convergence is 1.

b) First write

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(x+\frac{1}{2}\right)^n. \tag{2}$$

Therefore

$$a_0 = 0, \qquad a_n = \frac{2^n}{n^2} \text{ for all } n \geqslant 1. \tag{3}$$

Although $\frac{a_1}{a_0}$ is meaningless, as we are studying the limit, this doesn't matter. We compute

$$\rho^{-1} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}/(n+1)^2}{2^n/n^2}\right| = 2. \tag{4}$$

Therefore the radius of convergence is 1/2.

c) We have $a_n = \frac{n!}{n^n}$. So

$$\rho^{-1} = \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \frac{n!/(n+1)^n}{n!/n^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = e^{-1}. \tag{5}$$

So the radius of convergence is e.

Problem 2. Calculate the Taylor expansion for the following functions.

a) $\ln x$, $x_0 = 1$;

b) e^{x^3}, $x_0 = 0$;

c) $\frac{1}{1-x}$, $x_0 = 2$.

Solution.

a) We compute

$$(\ln x)' = x^{-1};\ (\ln x)'' = -x^{-2};\ (\ln x)''' = 2x^{-3};\ (\ln x)'''' = -6x^{-4} \tag{6}$$

and in general

$$(\ln x)^{(n)} = (-1)^{n-1}(n-1)!\,x^{-n}. \tag{7}$$

So

$$(\ln x)^{(n)}|_{x=x_0} = (-1)^{n-1}(n-1)! \tag{8}$$

and the Taylor expansion is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n. \tag{9}$$

b) Let $t=x^3$. We know

$$e^t=\sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{10}$$

holds for all t. Therefore

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \tag{11}$$

is the Taylor expansion of e^{x^3}.

c) We compute

$$\left(\frac{1}{1-x}\right)'=\frac{1}{(1-x)^2},\ldots,\left(\frac{1}{1-x}\right)^{(n)}=\frac{n!}{(1-x)^{n+1}}. \tag{12}$$

Therefore

$$\left(\frac{1}{1-x}\right)^{(n)}|_{x=x_0}=(-1)^{n+1}\,n! \tag{13}$$

and the Taylor expansion is

$$\sum_{n=0}^{\infty} (-1)^{n+1}\,(x-2)^n. \tag{14}$$

Alternative method: Let $t=x-2$, then $1-x=-t-1$. So we expand

$$\frac{1}{-t-1}\sim-\sum_{n=0}^{\infty} (-1)^n\,t^n\Longrightarrow\frac{1}{1-x}\sim\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n. \tag{15}$$

Problem 3. Rewrie the given expression as a sum whose generic term involves x^n:

a) $\sum_{n=0}^{\infty} a_n\,x^{n+2}$;

b) $(1-x^2)\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}$.

Solution.

a) Let $k=n+2$. Then

$$\sum_{n=0}^{\infty} a_n\,x^{n+2}=\sum_{k=2}^{\infty} a_{k-2}\,x^k=\sum_{n=2}^{\infty} a_{n-2}\,x^n. \tag{16}$$

b) We have

$$\begin{aligned}(1-x^2)\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} &= \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}-x^2\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}\\ &= \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}-\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n\\ &= \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n-\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n\\ &= 2\,a_2+6\,a_3\,x+\sum_{n=2}^{\infty} [(n+2)\,(n+1)\,a_{n+2}-n\,(n-1)\,a_n]\,x^n.\end{aligned} \tag{17}$$

Problem 4. Find the first five nonzero terms in the solution of the problem

$$y''-x\,y'-y=0,\qquad y(0)=2,\quad y'(0)=1. \tag{18}$$

Solution. Write

$$y=\sum_{n=0}^{\infty} a_n\,x^n. \tag{19}$$

Substitute into the equation:

$$0 = \left(\sum_{n=0}^{\infty} a_n\,x^n\right)''-x\left(\sum_{n=0}^{\infty} a_n\,x^n\right)'-\left(\sum_{n=0}^{\infty} a_n\,x^n\right) \tag{20}$$

$$= \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}-x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1}-\sum_{n=0}^{\infty} a_n\,x^n \tag{21}$$

$$= \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n-\sum_{n=1}^{\infty} n\,a_n\,x^n-\sum_{n=0}^{\infty} a_n\,x^n \tag{22}$$

$$= 2\,a_2-a_0+\sum_{n=1}^{\infty} [(n+2)\,(n+1)\,a_{n+2}-(n+1)\,a_n]\,x^n. \tag{23}$$

Thus the recurrence relations are

$$\begin{aligned} 2\,a_2 - a_0 &= 0, & (24)\\ (n+2)\,a_{n+2} - a_n &= 0. & (25)\end{aligned}$$

Now the initial conditions give

$$y(0) = 2 \Longrightarrow a_0 = 2; \qquad y'(0) = 1 \Longrightarrow a_1 = 1. \tag{26}$$

We compute

$$\begin{aligned} (n=0) \qquad a_2 &= \frac{a_0}{2} = 1; & (27)\\ (n=1) \qquad a_3 &= \frac{a_1}{3} = \frac{1}{3}; & (28)\\ (n=2) \qquad a_4 &= \frac{a_2}{4} = \frac{1}{4}. & (29)\end{aligned}$$

We already have five nonzero terms:

$$y(x) = 2 + x + x^2 + \frac{1}{3}\,x^3 + \frac{1}{4}\,x^4 + \cdots \tag{30}$$

Intermediate

Advanced

Problem 5. Use power series to solve

$$y'' - 2\,y' + y = 0. \tag{31}$$

Solution. Write

$$y = \sum_{n=0}^{\infty} a_n\,x^n \tag{32}$$

and substitute into the equation:

$$\sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,x^n - 2\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,x^n + \sum_{n=0}^{\infty} a_n\,x^n = 0. \tag{33}$$

The recurrence relation is

$$a_{n+2}\,(n+2)\,(n+1) - 2\,(n+1)\,a_{n+1} + a_n = 0. \tag{34}$$

or

$$a_{n+2} = \frac{2\,(n+1)\,a_{n+1} - a_n}{(n+2)\,(n+1)}. \tag{35}$$

We try to find a general formula for a_n. First shift the index

$$a_{n+2} = \frac{2\,(n+1)\,a_{n+1} - a_n}{(n+2)\,(n+1)} \Longrightarrow a_n = \frac{2\,(n-1)\,a_{n-1} - a_{n-2}}{n\,(n-1)}. \tag{36}$$

Now calculate for a few terms:

$$\begin{aligned} a_n = \frac{2\,(n-1)\,a_{n-1} - a_{n-2}}{n\,(n-1)} &= \frac{2}{n}\,a_{n-1} - \frac{1}{n(n-1)}\,a_{n-2}\\ &= \frac{2}{n}\left[\frac{2}{(n-1)}\,a_{n-2} - \frac{1}{(n-1)\,(n-2)}\,a_{n-3}\right] - \frac{1}{n(n-1)}\,a_{n-2}\\ &= \frac{3}{n\,(n-1)}\,a_{n-2} - \frac{2}{n\,(n-1)\,(n-2)}\,a_{n-3}\\ &= \frac{3}{n\,(n-1)}\left[\frac{2}{n-2}\,a_{n-3} - \frac{1}{(n-2)\,(n-3)}\,a_{n-4}\right] - \frac{2}{n\,(n-1)\,(n-2)}\,a_{n-3}\\ &= \frac{4}{n\,(n-1)\,(n-2)}\,a_{n-3} - \frac{3}{n\,(n-1)\,(n-2)\,(n-3)}\,a_{n-4}. & (37)\end{aligned}$$

Now we guess:

$$a_n = \frac{k+1}{n\,(n-1)\cdots(n-k+1)}\,a_{n-k} - \frac{k}{n\,(n-1)\cdots(n-k)}\,a_{n-k-1}. \tag{38}$$

To check whether this is true, compute one more term:

$$\begin{aligned} a_n &= \frac{k+1}{n(n-1)\cdots(n-k+1)}\, a_{n-k} - \frac{k}{n(n-1)\cdots(n-k)}\, a_{n-k-1} \\ &= \frac{k+1}{n(n-1)\cdots(n-k+1)}\left[\frac{2}{n-k}\, a_{n-k-1} - \frac{1}{(n-k)(n-k-1)}\, a_{n-k-2}\right] \\ &\quad - \frac{k}{n(n-1)\cdots(n-k)}\, a_{n-k-1} \\ &= \frac{k+2}{n(n-1)\cdots(n-k)}\, a_{n-k-k} - \frac{k+1}{n(n-1)\cdots(n-k-1)}\, a_{n-k-2} \end{aligned} \tag{39}$$

which is exactly what we get from replace k by $k+1$ in our formula.

Therefore setting $k=n-1$ we get (for $n \geqslant 2$)

$$a_n = \frac{n}{n(n-1)\cdots 2}\, a_1 - \frac{n-1}{n!}\, a_0 = \frac{1}{(n-1)!}\, a_1 - \frac{1}{(n-1)!}\, a_0 + \frac{1}{n!}\, a_0. \tag{40}$$

The solution is then

$$y = a_0 + a_1 x + \sum_{n=2}^{\infty}\left[\frac{1}{(n-1)!}\, a_1 - \frac{1}{(n-1)!}\, a_0 + \frac{1}{n!}\, a_0\right] x^n. \tag{41}$$

Simplify:

$$\begin{aligned} y &= a_0 + a_1 x + (a_1 - a_0)\sum_{n=2}^{\infty}\frac{x^n}{(n-1)!} + a_0 \sum_{n=2}^{\infty}\frac{x^n}{n!} \\ &= (a_1 - a_0)\, x + (a_1 - a_0)\sum_{n=2}^{\infty}\frac{x^n}{(n-1)!} + a_0 + a_0 x + a_0\sum_{n=2}^{\infty}\frac{x^n}{n!} \\ &= (a_1 - a_0)\, x\left[1 + \sum_{n=2}^{\infty}\frac{x^{n-1}}{(n-1)!}\right] + a_0\left[1 + x + \sum_{n=2}^{\infty}\frac{x^n}{n!}\right] \\ &= (a_1 - a_0)\, x\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{n!}\right] + a_0\left[1 + x + \sum_{n=2}^{\infty}\frac{x^n}{n!}\right] \\ &= (a_1 - a_0)\, x\sum_{n=0}^{\infty}\frac{x^n}{n!} + a_0\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= (a_1 - a_0)\, x\, e^x + a_0\, e^x. \end{aligned} \tag{42}$$

CHALLENGE

Problem 6. Consider two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ with radii of convergence ρ_1, ρ_2 respectively. We claim that the radius of convergence for the power series $\sum_{n=0}^{\infty} c_n x^n$ corresponding to the ratio $\sum_{n=0}^{\infty} a_n x^n / \sum_{n=0}^{\infty} b_n x^n$ is **at least**

$$\min\left(\text{Distance between 0 and the closest zero of } \sum_{n=0}^{\infty} b_n x^n,\ \ \rho_1,\ \ \rho_2\right). \tag{43}$$

Construct an example to show that "at least" cannot be dropped.

Solution. A simple example is

$$\frac{\sum_{n=1}^{\infty}\frac{x^n}{n}}{x}. \tag{44}$$