Oct. 14, 2011

Math 334 Fall 2011 Homework 5 Solutions

Basic

Intermediate

Problem 1. Solve the following equations.

a) $y'' + y = \csc^3 x$.

b) $y'' - 2\,y' + y = x^{-1}\,e^x$.

Solution.

a) We use variation of parameters.

- Write the equation in standard form (already done). Identify $g = \csc^3 x = \frac{1}{(\sin x)^3}$.
- Find y_1, y_2. Solve
$$y'' + y = 0 \tag{1}$$
we get
$$y_1 = \cos x, \quad y_2 = \sin x. \tag{2}$$
- Find y_p.
First compute the Wronskian:
$$W[y_1, y_2] = y_1\,y_2' - y_1'\,y_2 = 1. \tag{3}$$
Then compute
$$\begin{aligned} u_1 &= \int \frac{-g\,y_2}{W} \\ &= \int -\frac{1}{\sin^2 x}\,\mathrm{d}x \\ &= \frac{\cos x}{\sin x}. \end{aligned} \tag{4}$$
$$\begin{aligned} u_2 &= \int \frac{g\,y_1}{W} \\ &= \int \frac{\cos x}{(\sin x)^3}\,\mathrm{d}x \\ &= \int \frac{\mathrm{d}\sin x}{(\sin x)^3} \\ &= -\frac{1}{2}\,\frac{1}{\sin^2 x}. \end{aligned} \tag{5}$$
So
$$y_p = u_1\,y_1 + u_2\,y_2 = \frac{\cos x}{\sin x}\cos x - \frac{1}{2\sin^2 x}\sin x = \frac{\cos^2 x}{\sin x} - \frac{1}{2\sin x}. \tag{6}$$
Check solution: To do this we first see whether y_p can be simplified:
$$y_p = \frac{\cos^2 x}{\sin x} - \frac{1}{2\sin x} = \frac{1}{\sin x} - \sin x - \frac{1}{2\sin x} = \frac{1}{2\sin x} - \sin x. \tag{7}$$
As $\sin x$ solves the homogeneous equation, we only need to make sure that $\frac{1}{2\sin x}$ is a particular solution. Compute
$$\left(\frac{1}{2\sin x}\right)' = -\frac{1}{2}\,\frac{\cos x}{\sin^2 x}; \tag{8}$$
$$\left(\frac{1}{2\sin x}\right)'' = -\frac{1}{2}\left(\frac{\cos x}{\sin^2 x}\right)' = -\frac{1}{2}\,\frac{-\sin x\sin^2 x - 2\sin x\cos x\cos x}{\sin^4 x} = \frac{1}{2\sin x} + \frac{\cos^2 x}{\sin^3 x}. \tag{9}$$

Now

$$\left(\frac{1}{2\sin x}\right)'' + \frac{1}{2\sin x} = \frac{1}{\sin x} + \frac{\cos^2 x}{\sin^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin^3 x} = \frac{1}{\sin^3 x} = \csc^3 x. \tag{10}$$

- Write down solution.

$$y = C_1 \cos x + C_2 \sin x + \frac{\cos^2 x}{\sin x} - \frac{1}{2\sin x} \tag{11}$$

or equivalently

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2\sin x}. \tag{12}$$

b) The equation is already in standard form, with $g = x^{-1} e^x$.

- First solve

$$y'' - 2\,y' + y = 0 \tag{13}$$

to get

$$y_1 = e^x, \qquad y_2 = x\,e^x. \tag{14}$$

- Now compute the Wronskian:

$$W[y_1, y_2] = e^x\,(e^x + x\,e^x) - e^x\,(x\,e^x) = e^{2x}. \tag{15}$$

We have

$$\begin{aligned} u_1 &= \int \frac{-g\,y_2}{W} \\ &= \int \frac{-x^{-1} e^x\,x\,e^x}{e^{2x}}\,\mathrm{d}x \\ &= \int -\mathrm{d}x \\ &= -x. \end{aligned} \tag{16}$$

$$\begin{aligned} u_2 &= \int \frac{g\,y_1}{W} \\ &= \int \frac{x^{-1}\,e^x\,e^x}{e^{2x}}\,\mathrm{d}x \\ &= \ln|x|. \end{aligned} \tag{17}$$

So

$$y_p = -x\,e^x + \ln|x|\,x\,e^x \tag{18}$$

As $-x\,e^x$ solves the homogeneous equation, we can write

$$y_p = x\,e^x \ln|x|. \tag{19}$$

To save time in checking the solution, we write

$$y_p = \ln|x|\,y_2. \tag{20}$$

Now

$$y_p' = (\ln|x|)\,y_2' + x^{-1}\,y_2; \tag{21}$$

$$y_p'' = (\ln|x|)\,y_2'' + 2\,x^{-1}\,y_2' - x^{-2}\,y_2. \tag{22}$$

Now using the fact that $y_2'' - 2\,y_2' + y_2 = 0$, and $y_2 = x\,e^x$ so $y_2' = y_2 + e^x$, we have

$$y_p'' - 2\,y_p' + y_p = x^{-1}\,e^x. \tag{23}$$

- Write down the general solution:

$$y = C_1\,e^x + C_2\,x\,e^x + x\,e^x \ln|x|. \tag{24}$$

ADVANCED

Problem 2. Solve the following equations.

a) $y''' + 4\,y'' + y' - 6\,y = 0$

b) $y^{(4)} - 13\, y^{(2)} + 36\, y = 0.$

c) $y^{(5)} - y = 0.$

d) $y^{(7)} - 3\, y^{(6)} + 4\, y^{(5)} - 4\, y^{(4)} + 3\, y^{(3)} - y^{(2)} = 0.$

Solution.

a) The characteristic equation is

$$r^3 + 4\, r^2 + r - 6 = 0. \tag{25}$$

Notice that $r_1 = 1$ is a solution. So we factorize

$$r^3 + 4\, r^2 + r - 6 = (r - 1)\, (r^2 + 5\, r + 6). \tag{26}$$

So the other two solutions are $r_2 = -2, r_3 = -3$.

The general solution is then given by

$$y = C_1\, e^t + C_2\, e^{-2t} + C_3\, e^{-3t}. \tag{27}$$

b) Characteristic equation

$$r^4 - 13\, r^2 + 36 = 0. \tag{28}$$

Notice that only even powers appear. Let $u = r^2$. We have

$$u^2 - 13\, u + 36 = 0 \tag{29}$$

whose solutions are

$$u_1 = 9, u_2 = 4. \tag{30}$$

Now as $u = r^2$, we have

$$r_{1,2} = \pm 3, r_{3,4} = \pm 2. \tag{31}$$

Thus

$$y = C_1\, e^{3t} + C_2\, e^{-3t} + C_3\, e^{2t} + C_4\, e^{-2t}. \tag{32}$$

c) Characteristic equation:

$$r^5 - 1 = 0. \tag{33}$$

To solve it we need to write 1 into the form $R\, e^{i\theta}$. We have

$$R = |1| = 1. \tag{34}$$

θ is determined through

$$1 = 1\, (\cos\theta + i \sin\theta) \Longrightarrow \theta = 2\, k\, \pi \tag{35}$$

for all integers k.

Therefore

$$r = (1)^{1/5} = 1^{1/5}\, e^{i \frac{2k\pi}{5}} = e^{i \frac{2k\pi}{5}}. \tag{36}$$

Taking 5 consecutive k's, say $0, 1, -1, 2, -2$, we have

$$r_1 = e^{i0} = \cos 0 + i \sin 0 = 1; \tag{37}$$

$$r_2 = e^{i \frac{2\pi}{5}} = \cos \frac{2\,\pi}{5} + i \sin \frac{2\,\pi}{5}; \tag{38}$$

$$r_3 = e^{i \frac{-2\pi}{5}} = \cos \frac{2\,\pi}{5} - i \sin \frac{2\,\pi}{5}; \tag{39}$$

$$r_4 = \cos \frac{4\,\pi}{5} + i \sin \frac{4\,\pi}{5}; \tag{40}$$

$$r_5 = \cos \frac{4\,\pi}{5} - i \sin \frac{4\,\pi}{5}. \tag{41}$$

So the general solution is given by

$$y = C_1\, e^x + C_2\, e^{(\cos\frac{2}{5}\pi)t} \cos\left(\sin \frac{2\,\pi}{5}\, t\right) + C_3\, e^{(\cos\frac{2}{5}\pi)t} \sin\left(\sin \frac{2\,\pi}{5}\, t\right) + C_4\, e^{(\cos\frac{4}{5}\pi)t} \cos\left(\sin \frac{4\,\pi}{5}\, t\right) + C_5\, e^{(\cos\frac{4}{5}\pi)t} \sin\left(\sin \frac{4\,\pi}{5}\, t\right). \tag{42}$$

d) Characteristic equation:
$$r^7 - 3\,r^6 + 4\,r^5 - 4\,r^4 + 3\,r^3 - r^2 = 0. \tag{43}$$
First notice that $r_1 = r_2 = 0$. Factor out r^2, we get
$$r^5 - 3\,r^4 + 4\,r^3 - 4\,r^2 + 3\,r - 1 = 0. \tag{44}$$
Now notice that $r_3 = 1$ is a root. Factorize:
$$r^5 - 3\,r^4 + 4\,r^3 - 4\,r^2 + 3\,r - 1 = (r-1)\,(r^4 - 2\,r^3 + 2\,r^2 - 2\,r + 1). \tag{45}$$
Next notice that $r_4 = 1$ again solves
$$r^4 - 2\,r^3 + 2\,r^2 - 2\,r + 1 = 0 \tag{46}$$
so factorize
$$r^4 - 2\,r^3 + 2\,r^2 - 2\,r + 1 = (r-1)\,(r^3 - r^2 + r - 1). \tag{47}$$
Now it is clear that
$$r^3 - r^2 + r - 1 = (r-1)\,(r^2 + 1) \tag{48}$$
so
$$r_5 = 1, r_6 = i, r_7 = -i. \tag{49}$$
Summarize: We have 0 repeated twice, 1 repeated 3 times, then $\pm i$. So
$$y = C_1 + C_2\,t + C_3\,e^t + C_4\,t\,e^t + C_5\,t^2\,e^t + C_6 \cos t + C_7 \sin t. \tag{50}$$

CHALLENGE