Oct. 5, 2011

Math 334 Fall 2011 Homework 4 Solutions

Basic

Problem 1. Solve the following equations:

a) $3\,y''+8\,y'+4\,y=0$.

b) $y''+6\,y'+9\,y=0$.

c) $y''+2\,y'+10\,y=0$.

Solution.

a) Characteristic equation:

$$3\,r^2+8\,r+4=0 \tag{1}$$

Thus

$$r_{1,2}=\frac{-8\pm\sqrt{8^2-4\cdot 3\cdot 4}}{6}=\frac{-8\pm 4}{6}=-\frac{2}{3},-2. \tag{2}$$

So the general solution is given by

$$y=C_1\,e^{-2t/3}+C_2\,e^{-2t}. \tag{3}$$

b) Characteristic equation:

$$r^2+6\,r+9=0 \tag{4}$$

Thus

$$r_{1,2}=\frac{-6\pm\sqrt{6^2-4\cdot 9}}{2}=-3. \tag{5}$$

Repeated roots. So

$$y=C_1\,e^{-3t}+C_2\,t\,e^{-3t}. \tag{6}$$

c) Characteristic equation:

$$r^2+2\,r+10=0 \tag{7}$$

Thus

$$r_{1,2}=\frac{-2\pm\sqrt{2^2-4\cdot 10}}{2}=-1\pm 3\,i. \tag{8}$$

So the solution is given by

$$y=C_1\,e^{-t}\cos 3\,t+C_2\,e^{-t}\sin 3\,t. \tag{9}$$

Problem 2. Solve the following initial value problem.

a) $y''+3\,y'-4\,y=0,\ y(1)=0, y'(1)=1$.

b) $y''+2\,y'+4\,y=0,\ y(0)=1, y'(0)=1$.

Solution.

a) First find general solution. Characteristic equation:

$$r^2+3\,r-4=0\Longrightarrow r_{1,2}=-4,1. \tag{10}$$

So

$$y=C_1\,e^{-4t}+C_2\,e^t. \tag{11}$$

Now calculate

$$y'=-4\,C_1\,e^{-4t}+C_2\,e^t. \tag{12}$$

Thus

$$y(1)=0\Longrightarrow C_1\,e^{-4}+C_2\,e=0 \tag{13}$$

and

$$y'(1) = 1 \Longrightarrow -4\,C_1\,e^{-4} + C_2\,e = 1 \tag{14}$$

Multiply the first equation by 4 and add to the 2nd, we get $C_2 = \frac{e^{-1}}{5}$. Then C_1 can be obtained through either equation as $C_1 = -\frac{e^4}{5}$.

Therefore the solution is give by

$$y = -\frac{1}{5}\,e^{4-4t} + \frac{1}{5}\,e^{t-1}. \tag{15}$$

b) First find general solution:

$$r^2 + 2\,r + 4 = 0 \Longrightarrow r_{1,2} = -1 \pm \sqrt{3}\,i \tag{16}$$

so

$$y = C_1\,e^{-t}\cos\sqrt{3}\,t + C_2\,e^{-t}\sin\sqrt{3}\,t. \tag{17}$$

Calculate

$$y' = -C_1\,e^{-t}\cos\sqrt{3}\,t - \sqrt{3}\,C_1\,e^{-t}\sin\sqrt{3}\,t - C_2\,e^{-t}\sin\sqrt{3}\,t + \sqrt{3}\,C_2\,e^{-t}\cos\sqrt{3}\,t. \tag{18}$$

Using the initial conditions:

$$y(0) = 1 \Longrightarrow C_1 = 1; \tag{19}$$

$$y'(0) = 1 \Longrightarrow -C_1 + \sqrt{3}\,C_2 = 1. \tag{20}$$

We obtain

$$C_1 = 1, \quad C_2 = \frac{2}{\sqrt{3}}. \tag{21}$$

So the solution is given by

$$y = e^{-t}\cos\sqrt{3}\,t + \frac{2}{\sqrt{3}}\,e^{-t}\sin\sqrt{3}\,t. \tag{22}$$

INTERMEDIATE

Problem 3. Find the general solution for

$$y'' + 2\,y' + y = 2\,e^{-t}. \tag{23}$$

Solution.

We apply the method of undetermined coefficients.

First solve the corresponding homogeneous equation

$$y'' + 2\,y' + y = 0. \tag{24}$$

Characteristic equation $r^2 + 2\,r + 1 = 0$ gives repeated roots $r_{1,2} = -1$. So

$$y_1 = e^{-t}, y_2 = t\,e^{-t}. \tag{25}$$

Next guess the correct form of y_p:

$$g(t) = 2\,e^{-t} = e^{\alpha t}\,(a_0 + \cdots + a_n\,t^n) \tag{26}$$

with $\alpha = -1,\ n = 0$. So

$$y_p = t^s\,e^{-t}\,A_0. \tag{27}$$

To determine s, check $\alpha = -1$ is a repeated root of $r^2 + 2\,r + 1 = 0$ so we have to take $s = 2$. Thus our guess is

$$y_p = A_0\,t^2\,e^{-t}. \tag{28}$$

Substitute back into the equation:

$$\begin{aligned} 2\,e^{-t} &= y_p'' + 2\,y_p' + y_p \\ &= (A_0\,t^2\,e^{-t})'' + 2\,(A_0\,t^2\,e^{-t})' + (A_0\,t^2\,e^{-t}) \\ &= 2\,A_0\,e^{-t} - 4\,A_0\,t\,e^{-t} + A_0\,t^2\,e^{-t} + 4\,A_0\,t\,e^{-t} - 2\,A_0\,t^2\,e^{-t} + A_0\,t^2\,e^{-t} \\ &= 2\,A_0\,e^{-t}. \end{aligned}$$

So $A_0 = 1$ and therefore

$$y_p = t^2\,e^{-t}. \tag{29}$$

Finally the general solution is given by

$$y = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}. \tag{30}$$

Problem 4. Solve the initial value problem

$$y'' + 4y = t^2 + 3e^t, \qquad y(0) = 0, \quad y'(0) = 2. \tag{31}$$

Solution.

First solve the corresponding homogeneous equation

$$y'' + 4y = 0 \tag{32}$$

whose characteristic equation is $r^2 + 4 = 0 \Longrightarrow r_{1,2} = \pm 2i$ so

$$y_1 = \cos 2t, \, y_2 = \sin 2t. \tag{33}$$

Now we guess y_p. Note that $g(t) = t^2 + 3e^t$ is of neither type. However, t^2 and $3e^t$ are. So we write $g_1(t) = t^2$, $g_2(t) = 3e^t$, and guess

$$y_{p1} = t^s (A_0 + A_1 t + A_2 t^2); \qquad y_{p2} = t^s B e^t. \tag{34}$$

Note that g_1 and g_2 both correspond to the type $e^{\alpha t}(a_0 + \cdots + a_n t^n)$ with $\alpha = 0$ and $\alpha = 1$ respectively. Recall that the roots to the characteristic equation are $\pm 2i$ so neither α is a solution. Consequently $s = 0$ in both y_{p1} and y_{p2}:

$$y_{p1} = A_0 + A_1 t + A_2 t^2; \qquad y_{p2} = B e^t. \tag{35}$$

- Get y_{p1}. Substitute into the equation (with right hand side g_1):

$$\begin{aligned} t^2 &= y_{p1}'' + 4y_{p1} \\ &= 2A_2 + 4(A_0 + A_1 t + A_2 t^2) \\ &= (2A_2 + 4A_0) + (4A_1)t + 4A_2 t^2 \end{aligned} \tag{36}$$

  therefore

$$2A_2 + 4A_0 = 0; \quad 4A_1 = 0; \quad 4A_2 = 1. \tag{37}$$

  which gives

$$A_2 = \frac{1}{4}, \quad A_1 = 0, \quad A_0 = -\frac{1}{8}. \tag{38}$$

  So

$$y_{p1} = -\frac{1}{8} + \frac{t^2}{4}. \tag{39}$$

- Get y_{p2}: Substitute into the equation with right hand side g_2:

$$\begin{aligned} 3e^t &= y_{p2}'' + 4y_{p2} \\ &= Be^t + 4Be^t \\ &= 5Be^t \Longrightarrow B = \frac{3}{5}. \end{aligned} \tag{40}$$

  So

$$y_{p2} = \frac{3}{5} e^t. \tag{41}$$

Thus we get

$$y_p = y_{p1} + y_{p2} = -\frac{1}{8} + \frac{t^2}{4} + \frac{3}{5} e^t. \tag{42}$$

The general solution is

$$y = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{8} + \frac{t^2}{4} + \frac{3}{5} e^t. \tag{43}$$

To use the initial conditions, first calculate

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t + \frac{t}{2} + \frac{3}{5} e^t. \tag{44}$$

Now

$$y(0) = 0 \Longrightarrow C_1 - \frac{1}{8} + \frac{3}{5} = 0 \Longrightarrow C_1 = -\frac{19}{40}; \tag{45}$$

$$y'(0) = 2 \Longrightarrow 2C_2 + \frac{3}{5} = 2 \Longrightarrow C_2 = \frac{7}{10}. \tag{46}$$

Thus the final answer is

$$y = -\frac{19}{40} \cos 2t + \frac{7}{10} \sin 2t - \frac{1}{8} + \frac{t^2}{4} + \frac{3}{5} e^t. \tag{47}$$

ADVANCED

Problem 5. Consider $a y'' + b y' + c y = 0$ with a, b, c constants. Assume that $a r^2 + b r + c = 0$ has repeated root $r_1 = r_2$. Thus $y_1 = e^{r_1 t}$. Show that reduction of order always gives $y_2 = t y_1$.

Proof. Let $y_2 = v y_1$. Substitute into the equation:

$$0 = a (v y_1)'' + b (v y_1)' + c (v y_1) = v [a y_1'' + b y_1' + c y_1] + a y_1 v'' + [2 a y_1' + b y_1] v'. \tag{48}$$

As y_1 is a solution, the first $[\cdots]$ is zero. So

$$a y_1 v'' + [2 a y_1' + b y_1] v' = 0. \tag{49}$$

Now because $a r^2 + b r + c = 0$ has repeated root, necessarily $r_1 = \frac{-b}{2a}$. So $y_1' = \left(e^{-\frac{b}{2a} t}\right)' = -\frac{b}{2a} y_1$. Substitute into the above equation we have

$$2 a y_1' + b y_1 = 0 \tag{50}$$

so

$$a y_1 v'' = 0 \Longrightarrow v'' = 0 \Longrightarrow v = C_1 t + C_2. \tag{51}$$

Thus we can always take $y_2 = t y_1$. $\square$

CHALLENGE

Problem 6. Explain why the method of undetermined coefficients is not practical anymore when the coefficients are not constants.

Problem 7. Show that reduction of order always works. That is it always gives a y_2 that is linearly independent of y_1.

Proof. The method works by letting $y_2 = v y_1$ and the solve the following equation to get v:

$$a y_1 v'' + [2 a y' + b y_1] v' = 0. \tag{52}$$

Now as $y_1 \neq 0$, if y_2 is linearly dependent with y_1, then necessarily $v = \text{constant}$. In other words, the method does not work only when all solutions to the v equation are constants, that is the general solution to the v equation has to be

$$v = C_1. \tag{53}$$

However, as $a y_1 \neq 0$, the v equation is a second order equation whose general solution involves two arbitrary constants. $\square$