Sept. 23, 2011

Math 334 Fall 2011 Homework 2 Solutions

Basic

Problem 1. The d operator. Calculate

a) $\mathrm{d}(\sin x + y)$;

b) $\mathrm{d}\left(\frac{x\,y}{x^2+y^2}\right)$;

c) $\mathrm{d}(e^{x\,y})$;

Solution.

a) We have

$$\mathrm{d}(\sin x + y) = \frac{\partial(\sin x + y)}{\partial x}\,\mathrm{d}x + \frac{\partial(\sin x + y)}{\partial y}\,\mathrm{d}y = \cos x\,\mathrm{d}x + \mathrm{d}y. \tag{1}$$

b) We calculate

$$\frac{\partial\left(\frac{x\,y}{x^2+y^2}\right)}{\partial x} = \frac{\frac{\partial}{\partial x}(x\,y)\,(x^2+y^2) - x\,y\,\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} = \frac{y\,(x^2+y^2) - 2\,x^2\,y}{(x^2+y^2)^2} = \frac{y\,(y^2-x^2)}{(x^2+y^2)^2}; \tag{2}$$

Similarly we have

$$\frac{\partial\left(\frac{x\,y}{x^2+y^2}\right)}{\partial y} = \frac{x\,(x^2-y^2)}{(x^2+y^2)^2}. \tag{3}$$

So

$$\mathrm{d}\left(\frac{x\,y}{x^2+y^2}\right) = \frac{y\,(y^2-x^2)}{(x^2+y^2)^2}\,\mathrm{d}x + \frac{x\,(x^2-y^2)}{(x^2+y^2)^2}\,\mathrm{d}y. \tag{4}$$

c) We calculate

$$\mathrm{d}(e^{x\,y}) = \frac{\partial(e^{x\,y})}{\partial x}\,\mathrm{d}x + \frac{\partial(e^{x\,y})}{\partial y}\,\mathrm{d}y = y\,e^{x\,y}\,\mathrm{d}x + x\,e^{x\,y}\,\mathrm{d}y. \tag{5}$$

Problem 2. Solve the following exact equations.

a) $(6\,x\,y^2 + 4\,x^3\,y)\,\mathrm{d}x + (6\,x^2\,y + x^4 + e^y)\,\mathrm{d}y = 0.$

b) $\left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1\right)\mathrm{d}x + \left(\frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2}\right)\mathrm{d}y = 0.$

Solution.

a) We are told that it's exact, so all we need to do is to find u. Comparing

$$\int (6\,x\,y^2 + 4\,x^3\,y)\,\mathrm{d}x \text{ and } \int (6\,x^2\,y + x^4 + e^y)\,\mathrm{d}y, \tag{6}$$

we see that they are of similar difficulty. So it doesn't matter how we start.

Write

$$u(x,y) = \int (6\,x\,y^2 + 4\,x^3\,y)\,\mathrm{d}x + g(y) = 3\,x^2\,y^2 + x^4\,y + g(y). \tag{7}$$

Now compute

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3\,x^2\,y^2 + x^4\,y + g(y)) = \frac{\partial}{\partial y}(3\,x^2\,y^2) + \frac{\partial}{\partial y}(x^4\,y) + \frac{\partial}{\partial y}g(y) = 6\,x^2\,y + x^4 + g'(y). \tag{8}$$

Comparing with

$$N(x,y) = 6\,x^2\,y + x^4 + e^y \tag{9}$$

we see that $g'(y) = e^y$ which gives $g(y) = e^y$.

So $u(x,y) = 3\,x^2\,y^2 + x^4\,y + e^y$ and the general solution is

$$3\,x^2\,y^2 + x^4\,y + e^y = C. \tag{10}$$

b) Comparing

$$\int \left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1\right)\mathrm{d}x \text{ and } \int \left(\frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2}\right)\mathrm{d}y \tag{11}$$

we see that they are of similar difficulty. We start with

$$\begin{aligned} u(x,y) &= \int \left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1\right) \mathrm{d}x + g(y) \\ &= \int \frac{1}{y}\sin\frac{x}{y}\,\mathrm{d}x + \int \left(-\frac{y}{x^2}\right)\cos\frac{y}{x}\,\mathrm{d}x + \int 1\,\mathrm{d}x + g(y) \\ &= \int \sin\frac{x}{y}\,\mathrm{d}\frac{x}{y} + \int \cos\frac{y}{x}\,\mathrm{d}\left(\frac{y}{x}\right) + x + g(y) \\ &= -\cos\frac{x}{y} + \sin\frac{y}{x} + x + g(y). \end{aligned} \tag{12}$$

Now compute

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(-\cos\frac{x}{y}\right) + \frac{\partial}{\partial y}\left(\sin\frac{y}{x}\right) + g'(y) \\ &= \left(\sin\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \left(\cos\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) + g'(y) \\ &= -\frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{x}\cos\frac{y}{x} + g'(y). \end{aligned}$$

Comparing with

$$N(x,y) = \frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2} \tag{13}$$

we see that

$$g'(y) = \frac{1}{y^2} \text{ so can take } g(y) = -\frac{1}{y}. \tag{14}$$

Finally the general solution is given by

$$-\cos\frac{x}{y} + \sin\frac{y}{x} + x - \frac{1}{y} = C. \tag{15}$$

Problem 3. Solve the following linear equations.

a) Solve

$$y' = 1 + 3y\tan x. \tag{16}$$

b) Solve

$$y' = 2\,x\,y + x, \qquad y(1) = 1. \tag{17}$$

Solution.

a) Write

$$y' - (3\tan x)\,y = 1. \tag{18}$$

So $p(x) = -3\tan x$ (**note that negative sign!**) and $g(x) = 1$. The integrating factor is

$$\mu = e^{\int -3\tan x} = e^{-3\int \frac{\sin x}{\cos x}\mathrm{d}x} = e^{3\int \frac{\mathrm{d}\cos x}{\cos x}} = e^{3\ln|\cos x|} = (\cos x)^3.^1 \tag{19}$$

Multiply both sides by $(\cos x)^3$ we should reach

$$((\cos x)^3\,y)' = (\cos x)^3. \tag{20}$$

Check

$$((\cos x)^3\,y)' = (\cos x)^3\,y' - 3\sin x\,(\cos x)^2\,y = (\cos x)^3\,[y' - 3\tan x\,y]. \tag{21}$$

So we have found the correct integrating factor.

Now integrate:

$$(\cos x)^3\,y = \int (\cos x)^3\,\mathrm{d}x + C. \tag{22}$$

The trick now is to write

$$\int (\cos x)^3\,\mathrm{d}x = \int (1 - \sin^2 x)\,\mathrm{d}\sin x = \sin x - \frac{1}{3}\sin^3 x \tag{23}$$

1. Note that here rigorously speaking we should have $\mu = \cos^3 x$ when $\cos x > 0$ and $-\cos^3 x$ when $\cos x < 0$. But this rigorous approach will give us the same result.

So finally the general solution is

$$y = \frac{1}{\cos^3 x}\left(\sin x - \frac{1}{3}\sin^3 x + C\right). \tag{24}$$

b) Rewrite it as

$$y' - 2\,x\,y = x. \tag{25}$$

So the integrating factor is

$$\mu = e^{-\int 2x} = e^{-x^2}. \tag{26}$$

Multiplying both sides by μ we reach

$$(e^{-x^2}\,y)' = x\,e^{-x^2}. \tag{27}$$

Check

$$(e^{-x^2}\,y)' = e^{-x^2}\,y' - 2\,x\,e^{-x^2}\,y = e^{-x^2}\,[y' - 2\,x\,y]. \tag{28}$$

Integrate

$$(e^{-x^2}\,y) = \int x\,e^{-x^2}\,\mathrm{d}x + C = \frac{1}{2}\int e^{-x^2}\,\mathrm{d}x^2 + C = -\frac{1}{2}\,e^{-x^2} + C. \tag{29}$$

So finally

$$y = C\,e^{x^2} - \frac{1}{2}. \tag{30}$$

Since it's an initial value problem, we substitute $y(1) = 1$ into the above formula:

$$1 = y(1) = C\,e^1 - \frac{1}{2} \Longrightarrow C = \frac{3}{2\,e}. \tag{31}$$

So the solution to the IVP is

$$y = \frac{3}{2\,e}\,e^{x^2} - \frac{1}{2}. \tag{32}$$

Problem 4. Solve the following separable equations.

a) Solve

$$y' = -\frac{x\,e^x\,y^3}{y+1} \tag{33}$$

b) Solve

$$y' = (\tan x)\,(\tan y) \tag{34}$$

Solution.

a) We have

$$y' = -x\,e^x\,\frac{y^3}{y+1} \tag{35}$$

Divide both sides by $y^3/(y+1)$ we reach

$$\frac{(y+1)}{y^3}\,y' = -x\,e^x. \tag{36}$$

Integrate

$$\int \frac{y+1}{y^3}\,\mathrm{d}y = \int y^{-2}\,\mathrm{d}y + \int y^{-3}\,\mathrm{d}y = -y^{-1} - \frac{1}{2}\,y^{-2}; \tag{37}$$

$$\int -x\,e^x\,\mathrm{d}x = -\int x\,\mathrm{d}e^x = -x\,e^x + e^x. \tag{38}$$

So the general solution is given by

$$(x-1)\,e^x - \left(y^{-1} + \frac{1}{2}\,y^{-2}\right) = C. \tag{39}$$

At the end we have to add back the zeros of $y^3/(y+1)$. The only value of y that makes it 0 is $y = 0$. So we have another solution $y = 0$.

b) Divide both sides by $\tan y$:

$$\frac{\cos y}{\sin y}\,\mathrm{d}y = \frac{\sin x}{\cos x}\,\mathrm{d}x. \tag{40}$$

Integrate

$$\int \frac{\cos y}{\sin y}\,\mathrm{d}y = \int \frac{\mathrm{d}\sin y}{\sin y} = \ln|\sin y|. \tag{41}$$

Similarly

$$\int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\ln|\cos x|. \tag{42}$$

So general solution is

$$\ln|\sin y| + \ln|\cos x| = C \tag{43}$$

which is equivalent to

$$|(\sin y)\cos x| = e^C. \tag{44}$$

Renaming e^C by C and getting rid of the absolution value, we see that this formula is equivalent to

$$(\sin y)\,(\cos x) = C, \qquad C \neq 0. \tag{45}$$

Finally we add back the zeros of $\frac{\sin y}{\cos y}$, that is those $y_i = k\,\pi$ with $k = ..., -2, -1, 0, 1, 2, ...$

Now notice that if we allow $C = 0$, those constant solutions are already included. So the final compact form of our solution is

$$(\sin y)\,(\cos x) = C \tag{46}$$

with C an arbitrary constant.

Problem 5. Are the following equations exact?

a) $3\,(x^2 + y^2)\,\mathrm{d}x + x\,(x^2 + 3\,y^2 + 6\,y)\,\mathrm{d}y = 0.$

b) $y\,(2\,x - y + 2)\,\mathrm{d}x + 2\,(x - y)\,\mathrm{d}y = 0.$

Solution.

a) We have

$$M = 3\,(x^2 + y^2) \Longrightarrow \frac{\partial M}{\partial y} = 6\,y \tag{47}$$

and

$$N = x\,(x^2 + 3\,y^2 + 6\,y) \Longrightarrow \frac{\partial N}{\partial x} = 3\,x^2 + 3\,y^2 + 6\,y \tag{48}$$

So not exact.

b) We have

$$M = y\,(2\,x - y + 2) \Longrightarrow \frac{\partial M}{\partial y} = 2\,x - 2\,y + 2 \tag{49}$$

$$N = 2\,(x - y) \Longrightarrow \frac{\partial N}{\partial x} = 2 \tag{50}$$

So not exact.

INTERMEDIATE

Problem 6. Solve the following equations

a) $3\,(x^2 + y^2)\,\mathrm{d}x + x\,(x^2 + 3\,y^2 + 6\,y)\,\mathrm{d}y = 0.$

b) $y\,(2\,x - y + 2)\,\mathrm{d}x + 2\,(x - y)\,\mathrm{d}y = 0.$

Solution.

a) We have seen that it is not exact. So we need to find μ such that

$$M\,\frac{\partial \mu}{\partial y} - N\,\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu \tag{51}$$

or for this problem:

$$3\,(x^2 + y^2)\,\frac{\partial \mu}{\partial y} - x\,(x^2 + 3\,y^2 + 6\,y)\,\frac{\partial \mu}{\partial x} = 3(x^2 + y^2)\,\mu. \tag{52}$$

First guess $\mu = \mu(x)$:

$$-x\,(x^2 + 3\,y^2 + 6\,y)\,\mu'(x) = 3(x^2 + y^2)\,\mu. \tag{53}$$

Clearly won't work.

Next guess $\mu = \mu(y)$:

$$3\,(x^2 + y^2)\,\mu'(y) = 3(x^2 + y^2)\,\mu \Longrightarrow \mu' = \mu \tag{54}$$

and we can take $\mu = e^y$.

Multiply the equation by e^y:

$$3\,e^y\,(x^2 + y^2)\,\mathrm{d}x + x\,e^y\,(x^2 + 3\,y^2 + 6\,y)\,\mathrm{d}y = 0. \tag{55}$$

We check

$$\frac{\partial}{\partial y}[3\,e^y\,(x^2 + y^2)] = 3\,e^y\,(x^2 + y^2) + 6\,y\,e^y \tag{56}$$

and compare with

$$\frac{\partial}{\partial x}[x\,e^y\,(x^2 + 3\,y^2 + 6\,y)] = 3\,x^2\,e^y + e^y\,(3\,y^2 + 6\,y). \tag{57}$$

We see that they are the same so we have found the correct integrating factor.

Now integrate

$$3\,e^y\,(x^2 + y^2)\,\mathrm{d}x + x\,e^y\,(x^2 + 3\,y^2 + 6\,y)\,\mathrm{d}y = 0. \tag{58}$$

Comparing

$$\int 3\,e^y\,(x^2 + y^2)\,\mathrm{d}x \text{ and } \int x\,e^y\,(x^2 + 3\,y^2 + 6\,y)\,\mathrm{d}y \tag{59}$$

we see that the former is much eaiser. So write

$$u(x, y) = \int 3\,e^y\,(x^2 + y^2)\,\mathrm{d}x + g(y) = x^3\,e^y + 3\,x\,y^2\,e^y + g(y). \tag{60}$$

Compute

$$\frac{\partial u}{\partial y} = x^3\,e^y + 3\,x\,y^2\,e^y + 6\,x\,y\,e^y + g'(y) \tag{61}$$

and compare with $x\,e^y\,(x^2 + 3\,y^2 + 6\,y)$ we see that $g'(y) = 0$ so we take $g = 0$.

The solution is given by

$$x^3\,e^y + 3\,x\,y^2\,e^y = C. \tag{62}$$

Finally, note that as the integrating factor $\mu = e^y$ is never zero, the equation after multiplication of μ is equivalent to the original equation. So the solution to the original equation is also

$$x^3\,e^y + 3\,x\,y^2\,e^y = C. \tag{63}$$

b) As we already know that the equation is not exact, we try to find $\mu(x, y)$ solving

$$M\,\frac{\partial \mu}{\partial y} - N\,\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu \tag{64}$$

which becomes for this problem

$$y\,(2\,x - y + 2)\,\frac{\partial \mu}{\partial y} - 2\,(x - y)\,\frac{\partial \mu}{\partial x} = -2\,(x - y)\,\mu. \tag{65}$$

Try $\mu = \mu(x)$:

$$-2\,(x - y)\,\mu' = -2\,(x - y)\,\mu \Longrightarrow \mu' = \mu \tag{66}$$

and we take $\mu = e^x$.

Multiply the equation by μ we get

$$e^x\,y\,(2\,x - y + 2)\,\mathrm{d}x + 2\,e^x\,(x - y)\,\mathrm{d}y = 0. \tag{67}$$

Check

$$\frac{\partial}{\partial y} e^x\,y\,(2\,x - y + 2) = 2\,x\,e^x - 2\,y\,e^x + 2\,e^x \tag{68}$$

and

$$\frac{\partial}{\partial x}[2\,e^x\,(x - y)\,] = 2\,e^x\,(x - y) + 2\,e^x \tag{69}$$

and indeed the same.

Now comparing

$$\int e^x\,y\,(2\,x - y + 2)\,\mathrm{d}x \text{ and } \int 2\,e^x\,(x - y)\,\mathrm{d}y \tag{70}$$

we see that the latter is clearly easier. So write

$$u(x, y) = \int 2\,e^x\,(x - y)\,\mathrm{d}y + g(x) = 2\,e^x\,x\,y - e^x\,y^2 + g(x). \tag{71}$$

Comparing

$$\frac{\partial u}{\partial x} = 2\,e^x\,y + 2\,e^x\,x\,y - e^x\,y^2 + g'(x) \tag{72}$$

and $e^x\,y\,(2\,x - y + 2)$ we see that $g'(x) = 0$ so can take $g = 0$. So the general solution to the transformed equation is

$$2\,e^x\,x\,y - e^x\,y^2 = C. \tag{73}$$

As the transformed equation is obtained from the original one by multiplying e^x which is never zero, the general solution to the original equation is also

$$2\,e^x\,x\,y - e^x\,y^2 = C. \tag{74}$$

ADVANCED

Problem 7. Solve

$$y' + \frac{x}{y} + 2 = 0, \qquad y(0) = 1. \tag{75}$$

Solution. This is a homogeneous equation. So we let $v = y/x$. This gives $y' = x\,v' + v$ and the equation for v turns out to be

$$x\,v' + v + \frac{1}{v} + 2 = 0 \tag{76}$$

which simplifies to

$$x\,v' + \frac{(v+1)^2}{v} = 0 \Longrightarrow \frac{v}{(v+1)^2}\,v' = -\frac{1}{x}. \tag{77}$$

Integrate:

$$\int \frac{v\,\mathrm{d}v}{(v+1)^2} = \int \frac{\mathrm{d}v}{v+1} - \int \frac{\mathrm{d}v}{(v+1)^2} = \ln|v+1| + \frac{1}{v+1}; \qquad \int \left(-\frac{1}{x}\right)\mathrm{d}x = -\ln|x|. \tag{78}$$

So the solution reads

$$\ln|v+1| + \frac{1}{v+1} = -\ln|x| + C \tag{79}$$

together with the zeros of $\frac{(v+1)^2}{v}$ which is $v = -1$.

Back to y:

$$\ln\left|\frac{y}{x} + 1\right| + \frac{1}{(y/x)+1} = -\ln|x| + C, \qquad \frac{y}{x} = -1 \tag{80}$$

which simplify to

$$\ln|y+x| + \frac{x}{y+x} = C, \qquad y = -x. \tag{81}$$

Now use the initial value $y(0) = -1$. First note that $y = -x$ does not satisfy it. Next substitute this IV into the general solution formula we get

$$\ln|-1+0| + \frac{0}{-1+0} = C \Longrightarrow C = 0. \tag{82}$$

So the solution to the IVP is

$$\ln|y+x| + \frac{x}{y+x} = 0. \tag{83}$$

It cannot be simplified anymore.

CHALLENGE

Problem 8. Consider the general linear 1st order equation

$$y' + p(x)\,y = g(x). \tag{84}$$

Write it as $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ and show that it is exact only when $p(x) = 0$. Explore possible integrating factors using the general theory.

Solution. We have

$$y' + p(x)\,y = g(x) \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,y = g(x) \Longrightarrow \mathrm{d}y + [p(x)\,y - g(x)]\,\mathrm{d}x = 0 \Longrightarrow [p(x)\,y - g(x)]\,\mathrm{d}x + \mathrm{d}y = 0. \tag{85}$$

So

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[p(x)\,y - g(x)] = p(x); \qquad \frac{\partial N}{\partial x} = 0. \tag{86}$$

It is clear that the equation is exact only when $p=0$.

An integrating factor must satisfy

$$M\,\frac{\partial\mu}{\partial y}-N\,\frac{\partial\mu}{\partial x}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu \tag{87}$$

which means

$$[p(x)\,y-g(x)]\,\frac{\partial\mu}{\partial y}-\frac{\partial\mu}{\partial x}=-p(x)\,\mu. \tag{88}$$

- Guess $\mu=\mu(x)$. We reach

$$\mu'=p(x)\,\mu\Longrightarrow\mu=C\,e^{\int p}\text{ is a class of integrating factors.} \tag{89}$$

- Guess $\mu=\mu(y)$. We reach

$$[p(x)\,y-g(x)]\,\mu'=-p(x)\,\mu \tag{90}$$

  which has no solution that is independent of x.

- Guess $\mu=\mu(x\,y)$. We have $\frac{\partial\mu}{\partial y}=\mu'\,x$, $\frac{\partial\mu}{\partial x}=\mu'\,y$ so

$$([p(x)\,y-g(x)]\,x-y)\,\mu'=-p(x)\,\mu \tag{91}$$

  still doesn't work.

Warning: In what follows we try to find out a formula for all possible integrating factors. That is, are there any other integrating factors besides $C\,e^{\int p}$? Read on only if you are curious about this.

——————————— What's below is **not** related to the exams! ———————————

We can try to show that $\mu=C\,e^{\int p}$ is the only possible integrating factor, that is all integrating factors must be independent of y. Write

$$[p(x)\,y-g(x)]\,\frac{\partial\mu}{\partial y}-\frac{\partial\mu}{\partial x}=-p(x)\,\mu \tag{92}$$

as

$$[p(x)\,y-g(x)]\,\frac{\partial\mu}{\partial y}=\frac{\partial\mu}{\partial x}-p(x)\,\mu \tag{93}$$

Multiply both sides by $e^{-\int p}$ and let $Z(x,y)=e^{-\int p}\,\mu$. All we need to do is to show that Z is independent of y (in fact, since we know $\mu=C\,e^{\int p}$, Z must be a constant if our conjecture is true).

We have

$$[p(x)\,y-g(x)]\,\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial x}\Longrightarrow p(x)\,y-g(x)=\frac{Z_x}{Z_y}. \tag{94}$$

Note that $p(x), g(x)$ are just arbitrary functions of x, the above the equivalent to

$$\left(\frac{Z_x}{Z_y}\right)_{y\,y}=0. \tag{95}$$

But at this stage we realize that our claim ($Z=Z(x)$) cannot be true, as $Z=x\,y$ clearly satisfies the above equation.

So it seems there may indeed be other integrating factors than $C\,e^{\int p}$.

In fact, we can reach the above conclusion in the following much more straightforward way. Assume that the solution is given by

$$u(x,y)=C. \tag{96}$$

Then clearly $u(x,y)\,e^{\int p}$ is also an integrating factor.

Inspired by this, we can actually show that any integrating factor takes the form

$$\mu(x,y)=H(u)\,e^{\int p}. \tag{97}$$

To see this, notice that $H(u)$ is exactly Z. As Z satisfies $p(x)\,y-g(x)=\frac{Z_x}{Z_y}$, we have

$$\mathrm{d}Z=f(x,y)\,\mathrm{d}u. \tag{98}$$

This means, $Z(x,y)$ and $u(x,y)$ share level sets (that is if u is constant along a curve, Z is also constant along the same curve). In other words, $Z=H(u)$ for some single variable function H.