Dec. 2, 2011

Math 334 Fall 2011 Homework 11 Solutions

Basic

Problem 1. Transform the following initial value problem into an initial value problem for a system:

$$u'' + p(t)\, u' + q(t)\, u = g(t), \qquad u(0) = u_0, u'(0) = v_0. \tag{1}$$

Solution. Let $v = u'$. Then $v' = u''$ and the equation becomes

$$v' + p(t)\, v + q(t)\, u = g(t) \tag{2}$$

and the initial value becomes

$$u(0) = u_0, \quad v(0) = v_0. \tag{3}$$

The system we are looking for is then

$$\begin{aligned} v' &= -q(t)\, u - p(t)\, v + g(t) && (4)\\ u' &= v && (5) \end{aligned}$$

with initial values

$$u(0) = u_0, \quad v(0) = v_0. \tag{6}$$

Intermediate

Problem 2. Express the solution of the following initial value problem in terms of a convolution integral:

$$y'' + 4\, y' + 4\, y = g(t); \qquad y(0) = 2, y'(0) = -3. \tag{7}$$

Solution.

First transform the equation:

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2\, Y - s\, y(0) - y'(0) = s^2\, Y - 2\, s + 3; && (8)\\ \mathcal{L}\{y'\} &= s\, Y - y(0) = s\, Y - 2 && (9) \end{aligned}$$

Denoting $\mathcal{L}\{g\} = G(s)$, we have the transformed equation as

$$(s^2 + 4\, s + 4)\, Y = G(s) + 2\, s + 5. \tag{10}$$

So

$$Y = \frac{G(s)}{s^2 + 4\, s + 4} + \frac{2\, s + 5}{s^2 + 4\, s + 4}. \tag{11}$$

Now take inverses:

- $\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2 + 4\, s + 4}\right\}$. We use the convolution theorem:

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2 + 4\, s + 4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} * \mathcal{L}^{-1}\{G\} = (e^{-2t}\, t) * g = \int_0^t e^{-2(t-\tau)}\, (t - \tau)\, g(\tau)\, \mathrm{d}\tau. \tag{12}$$

- $\mathcal{L}^{-1}\left\{\frac{2\, s + 5}{s^2 + 4\, s + 4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2} + \frac{2}{(s+2)}\right\} = e^{-2t}\, t + 2\, e^{-2t}$.

So the final answer is

$$y = \int_0^t e^{-2(t-\tau)}\, (t - \tau)\, g(\tau)\, \mathrm{d}\tau + e^{-2t}\, (t + 2). \tag{13}$$

Problem 3. Express the solution of the following initial value problem in terms of a convolution integral:

$$y^{(4)} - y = g(t); \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{14}$$

Solution. Taking transform of the equation we obtain

$$(s^4 - 1)\, Y = G(s) \Longrightarrow Y = \frac{G(s)}{s^4 - 1}. \tag{15}$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4 - 1}\right\} * g. \tag{16}$$

We compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4-1}\right\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}-\frac{1}{s^2+1}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}-\frac{1}{s+1}-\frac{2}{s^2+1}\right\}=\frac{1}{4}\left[e^t-e^{-t}-2\sin t\right]. \tag{17}$$

So the answer is

$$y(t)=\frac{1}{4}\int_0^t \left[e^{(t-\tau)}-e^{-(t-\tau)}-2\sin\left(t-\tau\right)\right] g(\tau)\,\mathrm{d}\tau. \tag{18}$$

Problem 4. Find all eigenvalues and eigenvectors for

a) $A=\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$;

b) $A=\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Solution.

a) We have

$$\det\left(A-\lambda I\right)=\det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix}=\lambda^2+4\lambda+3. \tag{19}$$

Solving

$$\lambda^2+4\lambda+3=0\Longrightarrow\lambda_1=-3,\lambda_2=-1. \tag{20}$$

So eigenvalues are $-3,-1$.

- Eigenvectors corresponding to -3: We solve

$$\left(A-(-3)\,I\right)\boldsymbol{x}=0 \tag{21}$$

which becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0\Longrightarrow\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=a\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{22}$$

- Eigenvectors corresponding to -1: We solve

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0\Longrightarrow\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=a\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{23}$$

b) We have

$$\begin{aligned}\det\left(A-\lambda I\right) &= \det\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \\ &= (3-\lambda)\,(-\lambda)\,(3-\lambda)+2\cdot2\cdot4+2\cdot2\cdot4 \\ &\quad -4\,(-\lambda)\,4-2\cdot2\cdot(3-\lambda)-2\cdot2\cdot(3-\lambda) \\ &= -\lambda^3+6\lambda^2-9\lambda+16+16+16\lambda-12+4\lambda-12+4\lambda \\ &= -\lambda^3+6\lambda^2+15\lambda+8. \end{aligned} \tag{24}$$

Now we solve

$$-\lambda^3+6\lambda^2+15\lambda+8=0. \tag{25}$$

Observe: $\lambda_1=-1$ is a root. Factorize:

$$-\lambda^3+6\lambda^2+15\lambda+8=(\lambda+1)\,(-\lambda^2+7\lambda+8). \tag{26}$$

Now solve:

$$-\lambda^2+7\lambda+8=0\Longrightarrow\lambda_2=8,\lambda_3=-1. \tag{27}$$

So in fact we have two eigenvalues: $\lambda_1=\lambda_2=-1,\lambda_3=8$.

Next we find eigenvectors corresponding to -1. We need to solve

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

Note that the solutions are given by x_1,x_2,x_3 satisfying

$$2x_1+x_2+2x_3=0. \tag{29}$$

In other words the eigenvectors are all vectors satisfying this equation.

To get an explicit formula for eigenvectors, we write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2\,x_1 - 2\,x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}. \tag{30}$$

There are no restriction on x_1, x_2. Therefore the eigenvectors corresponding to -1 is given by

$$a \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}. \tag{31}$$

Remark. Keep in mind that for an eigenvalue, its eigenvectors are not "several single vectors", but a collection of infinitely many vectors. As a consequence, there are more than one way to represent them. For example, in the above we have shown that eigenvectors corresponding to -1 can be represented as

$$a \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}. \tag{32}$$

with a, b arbitrary constants. The same set of vectors can also be writtn as

$$a \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \tag{33}$$

To see that they indeed represent the same set of vectors, we check:

1. The former includes the latter: That is any vector in the form of the latter can be represented by the former.
$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}. \tag{34}$$

2. The latter includes the former:
$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = (-1) \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \tag{35}$$

Now we turn to the eigenvalue 8. We need to solve

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{36}$$

We use Gaussian elimination:

$$\begin{aligned} \begin{pmatrix} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{pmatrix} & \Longrightarrow \begin{pmatrix} -5 & 2 & 4 & 0 \\ 1 & -4 & 1 & 0 \\ 4 & 2 & -5 & 0 \end{pmatrix} \qquad \text{(Simplify the 2nd row)} \\ & \Longrightarrow \begin{pmatrix} 1 & -4 & 1 & 0 \\ -5 & 2 & 4 & 0 \\ 4 & 2 & -5 & 0 \end{pmatrix} \qquad \text{(Switch 1st and 2nd row)} \\ & \Longrightarrow \begin{pmatrix} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 18 & -9 & 0 \end{pmatrix} \qquad \text{(first row $\times 5$ add to 2nd; $\times(-4)$ add to 3rd)} \\ & \Longrightarrow \begin{pmatrix} 1 & -4 & 1 & 0 \\ 0 & -18 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ & \Longrightarrow \begin{pmatrix} 1 & -4 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So the system for x_1, x_2, x_3 is equivalent to

$$\begin{aligned} x_1 - 4\,x_2 + x_3 &= 0 && (37)\\ -2\,x_2 + x_3 &= 0 && (38)\end{aligned}$$

Represent x_1, x_2 by x_3:

$$\begin{aligned} x_1 &= x_3 && (39)\\ x_2 &= \frac{1}{2}\,x_3. && (40)\end{aligned}$$

This gives

$$\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix} x_3\\ \frac{x_3}{2}\\ x_3\end{pmatrix} = x_3 \begin{pmatrix} 1\\ 1/2\\ 1\end{pmatrix}. \tag{41}$$

So the eigenvectors corresponding to 8 are

$$a\begin{pmatrix} 1\\ 1/2\\ 1\end{pmatrix} \tag{42}$$

where a is an arbitrary number.

ADVANCED

Problem 5. Prove the basic properties of convolution:

- $f*g = g*f$;
- $f*(g_1 + g_2) = f*g_1 + f*g_2$;
- $(f*g)*h = f*(g*h)$;
- $f*0 = 0*f = 0$.

Proof.

- $f*g = g*f$. Recall definition:
$$f*g = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau. \tag{43}$$
Now do the change of variable:
$$t' = t - \tau \Longrightarrow \mathrm{d}\tau = -\mathrm{d}t' \tag{44}$$
and the integral becomes
$$\int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau = \int_t^0 f(t')\, g(t-t')(-\mathrm{d}t') = \int_0^t g(t-t')\, f(t')\, \mathrm{d}t' = g*f. \tag{45}$$
- We have
$f*(g_1 + g_2) = \int_0^t f(t-\tau)\ [g_1(\tau) + g_2(\tau)]\ \mathrm{d}\tau = \int_0^t f(t-\tau)\ g_1(\tau)\ \mathrm{d}\tau + \int_0^t f(t-\tau)\ g_2(\tau)\ \mathrm{d}\tau = f*g_1 + f*g_2.$ (46)
- Use definition:
$$\begin{aligned} (f*g)*h &= \int_0^t (f*g)(t-\tau)\, h(\tau)\, \mathrm{d}\tau\\ &= \int_0^t \left[\int_0^{t-\tau} f(t-\tau-s) g(s)\, \mathrm{d}s\right] h(\tau)\, \mathrm{d}\tau\\ &= \int_0^t \int_0^{t-\tau} f(t-\tau-s)\, g(s)\, h(\tau)\, \mathrm{d}s\, \mathrm{d}\tau. && (47)\end{aligned}$$
As we would like to pair g and h together, we have to write f as $f(t-t')$. So introduce $t' = s + \tau$ in the inner integral – Thus $\mathrm{d}s = \mathrm{d}t'$. Then we have
$$\begin{aligned} \int_0^t \left[\int_0^{t-\tau} f(t-\tau-s) g(s)\, \mathrm{d}s\right] h(\tau)\, \mathrm{d}\tau &= \int_0^t \left[\int_\tau^t f(t-t')\, g(t'-\tau)\, \mathrm{d}t'\right] h(\tau)\, \mathrm{d}\tau\\ &= \int_0^t \int_\tau^t f(t-t')\, g(t'-\tau)\, h(\tau)\, \mathrm{d}t'\, \mathrm{d}\tau. && (48)\end{aligned}$$

Now we switch the order of the integration. The domain of the integration is $0<\tau<t'<t$. So t' runs from 0 to t while τ from 0 to t'. Therefore

$$\begin{aligned}
\int_0^t \int_\tau^t f(t-t')\, g(t'-\tau)\, h(\tau)\, \mathrm{d}t'\, \mathrm{d}\tau &= \int_0^t \left[\int_0^{t'} f(t-t')\, g(t'-\tau)\, h(\tau)\, \mathrm{d}\tau\right] \mathrm{d}t' \\
&= \int_0^t f(t-t') \left[\int_0^{t'} g(t'-\tau)\, h(\tau)\, \mathrm{d}\tau\right] \mathrm{d}t' \\
&= \int_0^t f(t-t')\, (g*h)(t')\, \mathrm{d}t' \\
&= f*(g*h). \qquad (49)
\end{aligned}$$

- This one is trivial:

$$f*0 = \int_0^t f(t-\tau)\, 0\, \mathrm{d}\tau = 0. \qquad (50)$$

Note that, all the above can be easily proved by the property $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$. However, implicit in that approach is the assumption that $\mathcal{L}^{-1}\{\mathcal{L}\{f\}\} = f$ whose proof is actually not easy. $\square$

Challenge

Problem 6. Derive the formula $\mathcal{L}^{-1}\{e^{-as}\, F(s)\} = f(t-a)\, u(t-a)$ using convolution.

Proof. We have

$$\begin{aligned}
\mathcal{L}^{-1}\{e^{-as}\, F(s)\} &= \mathcal{L}^{-1}\{e^{-as}\}*\mathcal{L}^{-1}\{F(s)\} \\
&= \delta(t-a)*f(t) \\
&= \int_0^t f(t-\tau)\, \delta(\tau-a)\, \mathrm{d}t' \\
&= f(t-a)\, u(t-a).
\end{aligned}$$

The last step follows from the following observation: When $t<a$, $\tau-a<0$ and therefore in the integral $\delta(t'-a)=0$. $\square$

Problem 7. Recall that we can write any single linear homogeneous equation of order n into a 1st order system consisting of n equations. Show that the Wronskian of the latter is the same as the Wronskian of the former.

Proof. Let the n-th order equation be

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_n(t)\, y = 0. \qquad (51)$$

It can be written into a system of n first order equations

$$\dot{\boldsymbol{x}} = P(t)\, \boldsymbol{x} \qquad (52)$$

through setting

$$x_1 = y, \quad x_2 = y', \ldots, \quad x_n = y^{(n-1)}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, P(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ & & & \ddots & \\ & & & & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \cdots & -p_1(t) \end{pmatrix} \qquad (53)$$

The Wronskian for the n-th order equation reads:

$$\det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \qquad (54)$$

which becomes the Wronskian for the system after identifying

$$\boldsymbol{x}^{(i)} = \begin{pmatrix} y_i \\ y_i' \\ \vdots \\ y_i^{(n-1)} \end{pmatrix}. \tag{55}$$

□

Problem 8. Let W be the Wronskian of n solutions $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ to the system

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n \qquad (56)\\ \vdots \quad &\vdots \quad \vdots \\ \dot{x}_n &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n. \qquad (57) \end{aligned}$$

Prove that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = (p_{11}(t) + \cdots + p_{nn}(t))\, W. \tag{58}$$

Proof. From properties of determinants we have

$$\frac{\mathrm{d}\left(\det\begin{pmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix}\right)}{\mathrm{d}t} = \det\begin{pmatrix} \dot{x}_1^{(1)} & \cdots & \dot{x}_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ & \ddots & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix} + \cdots + \det\begin{pmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ \dot{x}_n^{(1)} & \cdots & \dot{x}_n^{(n)} \end{pmatrix} \tag{59}$$

Here we have used the following property: The derivative of a determinant is the sum of n determinants, each obtained by putting derivative on one single row (or one single column). This can be proved by using the ultimate definition of determinants:

$$\det(M) = \sum_{\sigma \in \text{All permutations of } \{1,\ldots,n\}} (\text{sign of } \sigma)\, m_{1\sigma(1)} \cdots m_{n\sigma(n)}. \tag{60}$$

or through definition of derivative (the $\lim_{\delta \to 0}$ one) and use the following property of determinants:

$$\det\begin{pmatrix} \vdots & & \vdots \\ a_1 + b_1 & \cdots & a_n + b_n \\ \vdots & & \vdots \end{pmatrix} = \det\begin{pmatrix} \vdots & & \vdots \\ a_1 & \cdots & a_n \\ \vdots & & \vdots \end{pmatrix} + \det\begin{pmatrix} \vdots & & \vdots \\ b_1 & \cdots & b_n \\ \vdots & & \vdots \end{pmatrix} \tag{61}$$

Now we have

$$\dot{x}_1^{(1)} = p_{11}(t)\, x_1^{(1)} + p_{12}(t)\, x_2^{(1)} + \cdots; \cdots; x_1^{(n)} = p_{11}(t)\, x_1^{(n)} + \cdots + p_{1n}(t)\, x_n^{(n)}. \tag{62}$$

Substituting into the first determinant and use the property

$$\det\begin{pmatrix} \vdots & & \vdots \\ a_1 + b_1 & \cdots & a_n + b_n \\ \vdots & & \vdots \end{pmatrix} = \det\begin{pmatrix} \vdots & & \vdots \\ a_1 & \cdots & a_n \\ \vdots & & \vdots \end{pmatrix} + \det\begin{pmatrix} \vdots & & \vdots \\ b_1 & \cdots & b_n \\ \vdots & & \vdots \end{pmatrix} \tag{63}$$

we have

$$\begin{aligned} \det\begin{pmatrix} \dot{x}_1^{(1)} & \cdots & \dot{x}_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ & \ddots & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix} &= \det\begin{pmatrix} p_{11}(t)\, x_1^{(1)} & \cdots & p_{11}(t)\, x_n^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ & \ddots & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix} + \det\begin{pmatrix} p_{12}(t)\, x_2^{(1)} & \cdots & p_{12}(t)\, x_2^{(n)} \\ x^{(1)} & \cdots & x_2^{(n)} \\ & \ddots & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix} \\ &\quad + \text{Terms similar to the 2nd one.} \end{aligned} \tag{64}$$

Now using the following property: If a matrix has one row a multiple of another, then the determinant is 0, we see that only the first one is not 0.

But the first one is simply

$$\det\begin{pmatrix} p_{11}(t)\, x_1^{(1)} & \cdots & p_{11}(t)\, x_n^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ & \ddots & \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix} = p_{11}(t) \det\begin{pmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix} = p_{11}(t)\, W. \tag{65}$$

Dealing with the rest similarly, we reach

$$\frac{\mathrm{d}W}{\mathrm{d}t} = (p_{11}(t) + \cdots + p_{n\,n}(t))\,W. \tag{66}$$

Remark. It's interesting that if we put derivative on each column and write

$$\frac{\mathrm{d}\left(\det\begin{pmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix}\right)}{\mathrm{d}t} = \det\begin{pmatrix} \dot{\boldsymbol{x}}^{(1)} & \cdots & \boldsymbol{x}^{(n)} \end{pmatrix} + \cdots + \det\begin{pmatrix} \boldsymbol{x}^{(1)} & \cdots & \dot{\boldsymbol{x}}^{(n)} \end{pmatrix} \tag{67}$$

and then use $\dot{\boldsymbol{x}}^{(1)} = P(t)\,\boldsymbol{x}^{(1)}$ and so on, we seem to get stuck. The philosophical reason for this difference between the row-by-row approach and column-by-column approach seems to be that, when doing the row-by-row approach we are using the fact that $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ are all solutions in each determinant, while when in the column-by-column approach, in each determinant in the right hand side, we only take advantage of one $\boldsymbol{x}^{(i)}$ being a solution. $\square$