Nov. 25, 2011

MATH 334 FALL 2011 HOMEWORK 11

BASIC

Problem 1. Transform the following initial value problem into an initial value problem for a system:

$$u'' + p(t)\, u' + q(t)\, u = g(t), \qquad u(0) = u_0, u'(0) = v_0. \tag{1}$$

INTERMEDIATE

Problem 2. Express the solution of the following initial value problem in terms of a convolution integral:

$$y'' + 4\, y' + 4\, y = g(t); \qquad y(0) = 2, y'(0) = -3. \tag{2}$$

Problem 3. Express the solution of the following initial value problem in terms of a convolution integral:

$$y^{(4)} - y = g(t); \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{3}$$

Problem 4. Find all eigenvalues and eigenvectors for

a) $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$;

b) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

ADVANCED

Problem 5. Prove the basic properties of convolution:

- $f * g = g * f$;
- $f * (g_1 + g_2) = f * g_1 + f * g_2$;
- $(f * g) * h = f * (g * h)$;
- $f * 0 = 0 * f = 0$.

CHALLENGE

Problem 6. Derive the formula $\mathcal{L}^{-1}\{e^{-as}\, F(s)\} = f(t-a)\, u(t-a)$ using convolution.

Problem 7. Recall that we can write any single linear homogeneous equation of order n into a 1st order system consisting of n equations. Show that the Wronskian of the latter is the same as the Wronskian of the former.

Problem 8. Let W be the Wronskian of n solutions $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ to the system

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n & (4) \\ \vdots \quad & \vdots \quad \vdots \\ \dot{x}_n &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n. & (5) \end{aligned}$$

Prove that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = (p_{11}(t) + \cdots + p_{nn}(t))\, W. \tag{6}$$

Answers:

- Problem 1:

$$\begin{aligned} v' &= -q(t)\, u - p(t)\, v + g(t) && (7) \\ u' &= v && (8) \end{aligned}$$

  with initial values

$$u(0) = u_0, \quad v(0) = v_0. \tag{9}$$

- Problem 2: $y = \int_0^t e^{-2(t-\tau)}\, (t-\tau)\, g(\tau)\, \mathrm{d}\tau + e^{-2t}\, (t+2)$.
- Problem 3: $y(t) = \frac{1}{4} \int_0^t \left[e^{(t-\tau)} - e^{-(t-\tau)} - 2\sin(t-\tau) \right] g(\tau)\, \mathrm{d}\tau$.
- Problem 4: We write the solution as (eigenvalue, eigenvector) pairs.

  a) $\left(-3, a \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right); \left(-1, a \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

  b) $\left(-1, a \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right); \left(8, a \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}\right)$.