Nov. 25, 2011

MATH 334 FALL 2011 HOMEWORK 10 SOLUTIONS

BASIC

Problem 1. Express the following function using the unit step function. And sketch their graphs.

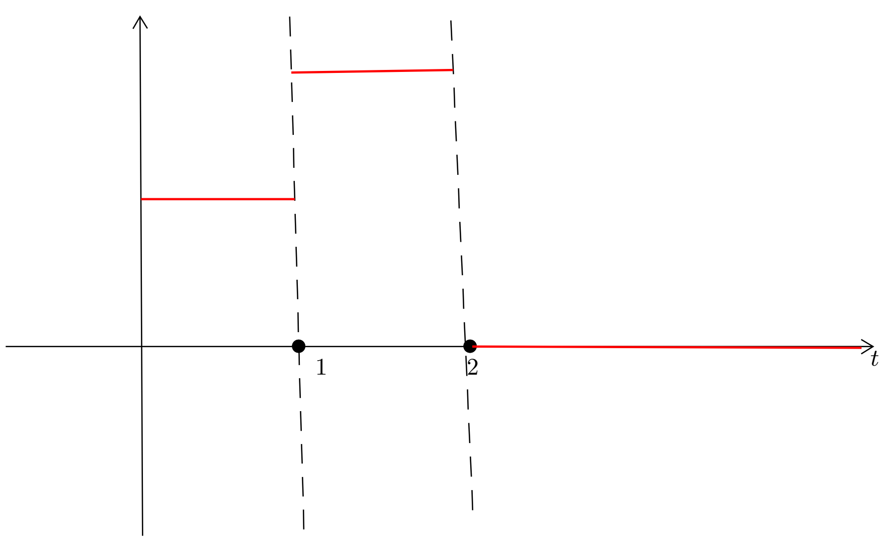
a) $g(t) = \begin{cases} 1 & 0 < t < 1 \\ 2 & 1 < t < 2 \\ 0 & t > 2 \end{cases}$

b) $g(t) = \begin{cases} t & t < 1 \\ t^2 & 1 < t < 2 \\ t^3 & t > 2 \end{cases}$

Solution.

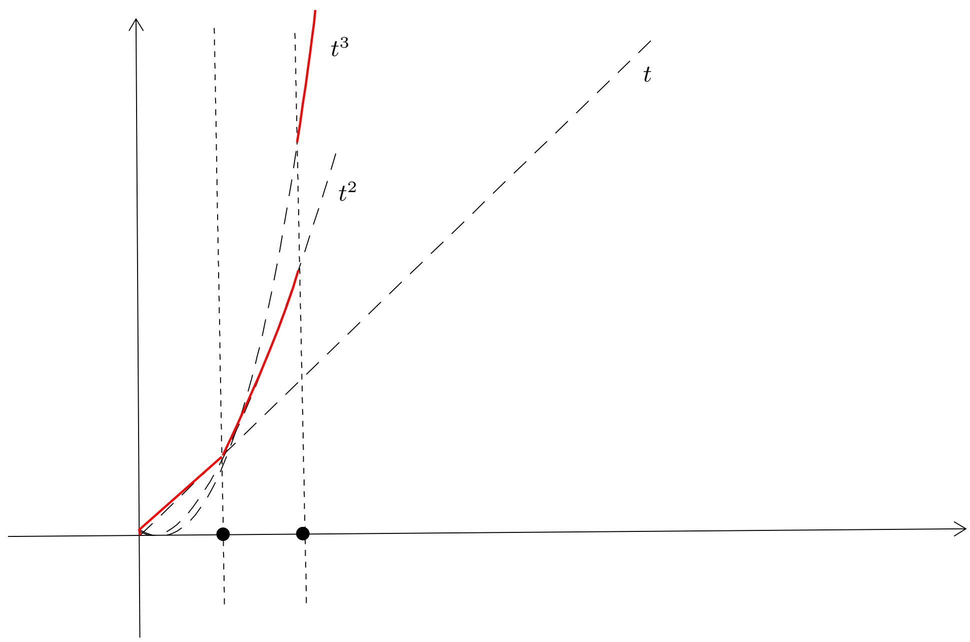
a) We have $g_1 = 1, g_2 = 2, g_3 = 0; t_1 = 1, t_2 = 2$. So

$$g(t) = 1 + (2 - 1)\, u(t - 1) + (0 - 2)\, u(t - 2) = 1 + u(t - 1) - 2\, u(t - 2). \tag{1}$$



b) We have $g_1 = t, g_2 = t^2, g_3 = t^3, t_1 = 1, t_2 = 2$. So

$$g(t) = t + (t^2 - t)\, u(t - 1) + (t^3 - t^2)\, u(t - 2). \tag{2}$$



Problem 2. Compute the following Laplace transforms:

a) $\mathcal{L}\{t\, u(t-2)\}$

b) $\mathcal{L}\big\{\cos 2t\, u\big(t-\frac{\pi}{8}\big) + (9\,t^2 + 2\,t - 1)\, u(t-2)\big\}$.

Solution.

a) Recall calculating $\mathcal{L}\{g(t)\, u(t-a)\}$:

1. Obtain $f(t) = g(t+a)$;
2. Compute $F(s) = \mathcal{L}\{f\}$.
3. Multiply it by $e^{-a s}$ to get $\mathcal{L}\{g(t)\, u(t-a)\} = e^{-a s} F(s)$.

We here have $g(t) = t, a = 2$. So first get

$$f(t) = g(t+2) = t + 2. \tag{3}$$

Now compute

$$F(s) = \mathcal{L}\{t+2\} = \frac{1}{s^2} + \frac{2}{s}. \tag{4}$$

Finally we have

$$\mathcal{L}\{t\, u(t-2)\} = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right). \tag{5}$$

b) We have to first use linearity:

$$\mathcal{L}\left\{\cos 2t\, u\left(t - \frac{\pi}{8}\right) + (9\,t^2 + 2\,t - 1)\, u(t-2)\right\} = \mathcal{L}\left\{\cos 2t\, u\left(t - \frac{\pi}{8}\right)\right\} + \mathcal{L}\{(9\,t^2 + 2\,t - 1)\, u(t-2)\}. \tag{6}$$

For the first transform we identify: $g(t) = \cos 2\,t, a = \frac{\pi}{8}$. So

$$f(t) = \cos\left(2\left(t + \frac{\pi}{8}\right)\right) = \cos\left(2\,t + \frac{\pi}{4}\right) = \cos(2\,t)\cos\frac{\pi}{4} - \sin(2\,t)\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(\cos(2\,t) - \sin(2\,t)). \tag{7}$$

Therefore

$$F(s) = \mathcal{L}\left\{\frac{\sqrt{2}}{2}(\cos(2\,t) - \sin(2\,t))\right\} = \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+4} - \frac{2}{s^2+4}\right) = \frac{\sqrt{2}}{2}\,\frac{s-2}{s^2+4}. \tag{8}$$

Finally

$$\mathcal{L}\left\{\cos 2t\, u\left(t - \frac{\pi}{8}\right)\right\} = e^{-\frac{\pi}{8}s}\,\frac{\sqrt{2}}{2}\,\frac{s-2}{s^2+4}. \tag{9}$$

Now the 2nd transform: We have $g(t) = 9\,t^2 + 2\,t - 1$, $a = 2$. So

$$f(t) = g(t+2) = 9\,(t+2)^2 + 2\,(t+2) - 1 = 9\,t^2 + 38\,t + 39. \tag{10}$$

We have

$$F(s) = \mathcal{L}\{9\,t^2 + 38\,t + 39\} = \frac{18}{s^3} + \frac{38}{s^2} + \frac{39}{s}. \tag{11}$$

So

$$\mathcal{L}\{(9\,t^2 + 2\,t - 1)\,u(t-2)\} = e^{-2s}\left(\frac{18}{s^3} + \frac{38}{s^2} + \frac{39}{s}\right). \tag{12}$$

Putting things together:

$$\mathcal{L}\left\{\cos 2t\, u\!\left(t - \frac{\pi}{8}\right) + (9\,t^2 + 2\,t - 1)\,u(t-2)\right\} = \frac{\sqrt{2}}{2}\,e^{-\frac{\pi}{8}s}\,\frac{s-2}{s^2+4} + e^{-2s}\left(\frac{18}{s^3} + \frac{38}{s^2} + \frac{39}{s}\right). \tag{13}$$

Problem 3. Compute $\mathcal{L}\{\cos(e^{t^2-1})\,\delta(t-1)\}$.
Solution. Recall

$$\mathcal{L}\{f(t)\,\delta(t-a)\} = f(a)\,e^{-a\,s}. \tag{14}$$

Here $f(t) = \cos(e^{t^2-1})$ and $a = 1$. So the answer is

$$e^{-s}\cos(e^{1^2-1}) = (\cos 1)\,e^{-s}. \tag{15}$$

Intermediate

Problem 4. Find the inverse Laplace transform for the following functions.

a) $F(s) = \frac{2\,(s-1)\,e^{-2s}}{s^2 - 2\,s + 2}$.

b) $F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$.

Solution.

a) Spotting e^{-2s} we know that the step function is involved. We use the formula

$$\mathcal{L}^{-1}\{e^{-a\,s}\,F(s)\} = f(t-a)\,u(t-a). \tag{16}$$

Here $a = 2$, $F(s) = \frac{2\,(s-1)}{(s^2 - 2\,s + 2)}$. We compute

$$f(t) = \mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{2\,(s-1)}{(s-1)^2 + 1}\right\} = 2\,e^t\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = 2\,e^t\cos t. \tag{17}$$

Therefore

$$f(t-2) = 2\,e^{t-2}\cos(t-2). \tag{18}$$

Finally

$$\mathcal{L}^{-1}\{F\} = 2\,e^{t-2}\cos(t-2)\,u(t-2). \tag{19}$$

b) We have

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} = u(t-1) + u(t-2) - u(t-3) - u(t-4). \tag{20}$$

Problem 5. Solve

$$y'' + y = g(t) = \begin{cases} t/2 & 0 \leqslant t < 6 \\ 3 & t \geqslant 6 \end{cases}, \qquad y(0) = 0, \quad y'(0) = 1. \tag{21}$$

Solution. First write

$$g(t) = t/2 + (3 - t/2)\,u(t-6). \tag{22}$$

Now compute

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y - 1. \tag{23}$$

$$\mathcal{L}\{g\} = \mathcal{L}\{t/2\} + \mathcal{L}\{(3 - t/2)\,u(t-6)\} = \frac{1}{2\,s^2} + e^{-6s}\,\mathcal{L}\left\{3 - \frac{t+6}{2}\right\} = \frac{1}{2\,s^2} - \frac{e^{-6s}}{2\,s^2}. \tag{24}$$

The transformed equation is then

$$s^2\,Y + Y = \frac{1}{2\,s^2} - \frac{e^{-6s}}{2\,s^2} + 1 \Longrightarrow Y = \frac{1}{2\,s^2\,(s^2+1)} - \frac{e^{-6s}}{2\,s^2\,(s^2+1)} + \frac{1}{s^2+1}. \tag{25}$$

To compute y, we compute the inverse Laplace transform of the right hand side one by one.

- $\mathcal{L}^{-1}\left\{\frac{1}{2\,s^2\,(s^2+1)}\right\}$. We use partial fraction.[1] Write

$$\frac{1}{2\,s^2\,(s^2+1)} = \frac{1/2}{s^2\,(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C\,s+D}{s^2+1} = \frac{A\,s\,(s^2+1) + B\,(s^2+1) + (C\,s+D)\,s^2}{s^2\,(s^2+1).}. \tag{26}$$

Thus we have

$$\frac{1}{2} = (A+C)s^3 + (B+D)s^2 + A\,s + B \tag{27}$$

and

$$A+C = 0 \tag{28}$$
$$B+D = 0 \tag{29}$$
$$A = 0 \tag{30}$$
$$B = \frac{1}{2}. \tag{31}$$

Consequently

$$A = C = 0, B = \frac{1}{2}, D = -\frac{1}{2}. \tag{32}$$

So the partial fraction representation is

$$\frac{1}{2\,s^2\,(s^2+1)} = \frac{1}{2\,s^2} - \frac{1}{2\,(s^2+1)}. \tag{33}$$

Consequently

$$\mathcal{L}^{-1}\left\{\frac{1}{2\,s^2\,(s^2+1)}\right\} = \frac{t}{2} - \frac{1}{2}\sin t. \tag{34}$$

- $\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{2\,s^2\,(s^2+1)}\right\}$. Recall the formula

$$\mathcal{L}^{-1}\{e^{-a\,s}\,F(s)\} = f(t-a)\,u(t-a). \tag{35}$$

Here $a = 6$, $F = \frac{1}{2\,s^2\,(s^2+1)}$. As we have just computed

$$f = \mathcal{L}^{-1}\left\{\frac{1}{2\,s^2\,(s^2+1)}\right\} = \frac{t}{2} - \frac{1}{2}\sin t \tag{36}$$

We can immediately write down

$$\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{2\,s^2\,(s^2+1)}\right\} = \left[\frac{t}{2} - 3 - \frac{1}{2}\sin\,(t-6)\right] u(t-6). \tag{37}$$

- The last term is standard:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t. \tag{38}$$

Putting everything together we have

$$y = \frac{t}{2} + \frac{1}{2}\sin t - \left[\frac{t}{2} - 3 - \frac{1}{2}\sin\,(t-6)\right] u(t-6). \tag{39}$$

Problem 6. Solve

$$y'' + y = \delta(t - 2\,\pi)\cos t, \qquad y(0) = 0, y'(0) = 1. \tag{40}$$

Solution. First transform the equation:

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y - 1; \qquad \mathcal{L}\{\delta(t-2\,\pi)\cos t\} = e^{-2\pi s}\cos\,(2\,\pi) = e^{-2\pi s}. \tag{41}$$

So the transformed equation is

$$(s^2+1)\,Y - 1 = e^{-2\pi s}. \tag{42}$$

This gives

$$Y = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}. \tag{43}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$.

1. In this problem it is possible to skip the following calculation by realizing $1/s^2 - 1/(s^2+1) = 1/[s^2\,(s^2+1)]$.

- $\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} = f(t-2\pi)\, u(t-2\pi)$ with

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t. \tag{44}$$

  So

$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} = u(t-2\pi)\sin(t-2\pi) = u(t-2\pi)\sin(t). \tag{45}$$

Summarizing, we have

$$y = \sin t\,[1 + u(t-2\pi)]. \tag{46}$$

Advanced

Problem 7. Let f satisfy $f(t+T) = f(t)$ for all $t \geqslant 0$ and some fixed positive number T. Show that

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,\mathrm{d}t}{1-e^{-sT}}. \tag{47}$$

Proof. We have

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\ &= \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_T^{2T} e^{-st} f(t)\,\mathrm{d}t + \cdots \\ &= \sum_{k=0}^\infty \int_{kT}^{(k+1)T} e^{-st} f(t)\,\mathrm{d}t. \end{aligned} \tag{48}$$

Now for each integral, do change of variable: $t' = t - kT$. We have

$$\int_{kT}^{(k+1)T} e^{-st} f(t)\,\mathrm{d}t = \int_0^T e^{-s(t'+kT)} f(t'+kT)\,\mathrm{d}t' = e^{-skT}\int_0^T e^{-st'} f(t')\,\mathrm{d}t' = e^{-skT}\int_0^T e^{-st} f(t)\,\mathrm{d}t. \tag{49}$$

Substitute back:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \sum_{k=0}^\infty \int_{kT}^{(k+1)T} e^{-st} f(t)\,\mathrm{d}t \\ &= \sum_{k=0}^\infty e^{-skT}\int_0^T e^{-st} f(t)\,\mathrm{d}t. \\ &= \left(\sum_{k=0}^\infty e^{-skT}\right)\int_0^T e^{-st} f(t)\,\mathrm{d}t. \\ &= \left(\sum_{k=0}^\infty (e^{-sT})^k\right)\int_0^T e^{-st} f(t)\,\mathrm{d}t \\ &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,\mathrm{d}t. \end{aligned}$$

Thus ends the proof. □

Challenge

Problem 8. Let $f(t)$ be a bounded function (not necessarily continuous). Prove that its Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \tag{50}$$

is continuous at all $s > 0$.[2] Therefore usually there is no need to consider the inverse transform of functions with jumps.

Proof. Let $M > 0$ be such that $|f(t)| \leqslant M$ for all t. Take any $s_0 > 0$, we show that $F(s)$ is continuous there. In other words, we show

$$\left|\int_0^\infty e^{-st} f(t)\,\mathrm{d}t - \int_0^\infty e^{-s_0 t} f(t)\mathrm{d}t\right| \longrightarrow 0 \tag{51}$$

2. This can be replaced by "continuous in its domain", but it seems the proof will become much more technical.

as $s \longrightarrow s_0$. We use the $\varepsilon - \delta$ formality. That is we show for any $\varepsilon > 0$, there is $\delta > 0$ such that when $|s - s_0| < \delta$,

$$\left| \int_0^\infty e^{-st} f(t)\,\mathrm{d}t - \int_0^\infty e^{-s_0 t} f(t)\mathrm{d}t \right| < \varepsilon. \tag{52}$$

Now for any $\varepsilon > 0$, there is $R > 0$ such that

$$\int_R^\infty e^{-s_0 t/2}\,\mathrm{d}t < \frac{\varepsilon}{4\,M}. \tag{53}$$

For such R we have, for all $|s - s_0| < s_0/2$,

$$\begin{aligned} \left| \int_R^\infty e^{-st} f(t)\,\mathrm{d}t - \int_R^\infty e^{-s_0 t} f(t)\mathrm{d}t \right| &\leqslant \left| \int_R^\infty e^{-st} f(t)\,\mathrm{d}t \right| + \left| \int_R^\infty e^{-s_0 t} f(t)\mathrm{d}t \right| \\ &\leqslant M \left| \int_R^\infty e^{-st}\,\mathrm{d}t \right| + M \left| \int_R^\infty e^{-s_0 t}\,\mathrm{d}t \right| \\ &\leqslant M \left[2 \int_R^\infty e^{-s_0 t/2}\,\mathrm{d}t \right] \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{54}$$

Next take δ' such that when $|s - s_0| < \delta'$,

$$|e^{-st} - e^{-s_0 t}| < \frac{\varepsilon}{2\,M R} \qquad \text{for all } 0 < t < R. \tag{55}$$

This leads to, when $|s - s_0| < \delta'$,

$$\begin{aligned} \left| \int_0^R e^{-st} f(t)\,\mathrm{d}t - \int_0^R e^{-s_0 t} f(t)\mathrm{d}t \right| &= \left| \int_0^R (e^{-st} - e^{-s_0 t})\, f(t)\,\mathrm{d}t \right| \\ &\leqslant M \int_0^R |e^{-st} - e^{-s_0 t}|\mathrm{d}t \\ &< M \frac{\varepsilon}{2\,M R} \int_0^R \mathrm{d}t = \frac{\varepsilon}{2}. \end{aligned}$$

Finally let $\delta = \min\{\delta', s_0/2\}$. We see that when $|s - s_0| < \delta$,

$$\left| \int_0^\infty e^{-st} f(t)\,\mathrm{d}t - \int_0^\infty e^{-s_0 t} f(t)\mathrm{d}t \right| \leqslant \left| \int_R^\infty e^{-st} f(t)\,\mathrm{d}t - \int_R^\infty e^{-s_0 t} f(t)\mathrm{d}t \right| + \left| \int_0^R e^{-st} f(t)\,\mathrm{d}t - \int_0^R e^{-s_0 t} f(t)\mathrm{d}t \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{56}$$

Thus ends the proof. □

Remark. Thanks to Delyle, I realized that the above proof is unnecessarily complicated. Following is a simpler one.

Let s_0 be fixed. We will show that $|F(s) - F(s_0)| \leqslant \frac{4\,M}{s_0^2}\,|s - s_0|$ for all s such that $|s - s_0| < s_0/2$. Continuity then follows naturally. Here M is as defined at the beginning of the above proof.

For such s we have (using mean value theorem: $f(a) - f(b) = f'(\xi)\,(a - b)$ with ξ between a and b)

$$|e^{-st} - e^{-s_0 t}| = |-t\,e^{-\xi t}\,(s - s_0)| \leqslant t\,e^{-s_0 t/2}\,|s - s_0|. \tag{57}$$

The last inequality comes from the fact that if ξ lies between s and s_0, then necessarily $\xi > s_0/2$.

Now we have

$$\left| \int_0^\infty e^{-st} f(t)\,\mathrm{d}t - \int_0^\infty e^{-s_0 t} f(t)\mathrm{d}t \right| \leqslant \int_0^\infty t\,e^{-s_0 t/2}\,|s - s_0|\,M\,\mathrm{d}t = |s - s_0| \left[M \int_0^\infty t\,e^{-s_0 t/2}\,\mathrm{d}t \right] = \frac{4\,M}{s_0^2}\,|s - s_0|. \tag{58}$$