SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

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1. Power series solutions.

1.1. An example.

So far we can effectively solve linear equations (homogeneous and non-homongeneous) with constant coefficients, but for equations with variable coefficients only special cases are discussed (1st order, etc.). Now we turn to this latter case and try to find a general method. The idea is to assume that the unknown function y can be expanded into a power series:

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

We try to determine the coefficients $a_0, a_1, \ldots$

Example 1. Solve

$$y' - 2 x y = 0. \tag{2}$$

Solution. Substitute

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{3}$$

into the equation. We have

$$\left(a_1 + 2 a_2 x + 3 a_3 x^2 + \cdots\right) - 2\left(a_0 x + a_1 x^2 + a_2 x^3 + \cdot\right) = 0. \tag{4}$$

Rewrite it to

$$a_1 + (2 a_2 - 2 a_0) x + (3 a_3 - 2 a_1) x^2 + \cdots = 0. \tag{5}$$

Naturally we require the coefficients to each power of x to be 0:

$$\begin{aligned} a_1 &= 0 && (6)\\ 2 a_2 - 2 a_0 &= 0 && (7)\\ 3 a_3 - 2 a_1 &= 0 && (8)\\ \vdots \quad & \quad \vdots \end{aligned}$$

We conclude

$$a_1 = 0, \; a_2 = a_0, \; a_3 = 0, \ldots \tag{9}$$

We see that the coefficients can be determined one by one.

However, as there are infinitely many a_i's, we need a general formula. To do this, we return to

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{10}$$

and write it as

$$y(x)=\sum_{n=0}^{\infty} a_n\,x^n. \tag{11}$$

Put this into the equation

$$y'-2\,x\,y=0 \tag{12}$$

we have

$$\begin{aligned} 0 &= \left(\sum_{n=0}^{\infty} a_n\,x^n\right)'-2\,x\left(\sum_{n=0}^{\infty} a_n\,x^n\right) \\ &= \sum_{n=1}^{\infty} (n\,a_n)\,x^{n-1}-\sum_{n=0}^{\infty} 2\,a_n\,x^{n+1} \\ &= \sum_{n=0}^{\infty} ((n+1)\,a_{n+1})\,x^n-\sum_{n=1}^{\infty} 2\,a_{n-1}\,x^n \\ &= a_1+\sum_{n=1}^{\infty} [(n+1)\,a_{n+1}-2\,a_{n-1}]\,x^n. \end{aligned} \tag{13}$$

As a consequence, we have

$$a_1=0, \qquad a_{n+1}=\frac{2}{n+1}\,a_{n-1}. \tag{14}$$

From this we clearly see that $a_{2k+1}=0$ for all k. On the other hand,

$$a_{2k}=\frac{2}{2\,k}\,a_{2k-2}=\frac{1}{k}\,a_{2k-2}=\cdots=\frac{1}{k!}\,a_0. \tag{15}$$

Therefore we have

$$y(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\,a_0\,x^{2k}=a_0\left[\sum_{k=0}^{\infty}\frac{1}{k!}\,x^{2k}\right]. \tag{16}$$

Now we recognize that

$$\sum_{k=0}^{\infty}\frac{1}{k!}\,x^{2k}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(x^2\right)^k=e^{x^2}. \tag{17}$$

So finally we have

$$y(x)=a_0\,e^{x^2} \tag{18}$$

where a_0 can take any value – recall that the general solution to a first order linear equation involves an arbitrary constant!

From this example we see that the method have the following steps:

1. Write

$$y(x)=\sum_{n=0}^{\infty} a_n\,x^n. \tag{19}$$

2. Substitute into the equation and determine a_n. A recurrence relation – a formula determining a_n using a_i, $i<n$ – is preferred.
3. Try to sum back and find out a closed form formula for y.

There are several theoretical issues we need to settle.

1. When we substitute $y(x)=\sum a_n\,x^n$ into the equation, we write

$$y'=\sum a_n\,(x^n)' \tag{20}$$

which is equivalent to claiming that it's OK to differentiate term by term:

$$\left(\sum a_n\,x^n\right)'=\sum (a_n\,x^n)' \tag{21}$$

2. We determine a_n by settle the coefficients of each x^n to 0. In other words, we claim that

$$\sum_{n=0}^{\infty} a_n x^n = 0 \Longleftrightarrow a_n = 0 \text{ for each } n. \tag{22}$$

3. In practice, it may happen that we cannot "sum back". Then there are two issues:

   a. Check whether $a_0 + a_1 x + \cdots$ is indeed a well-defined function.

   b. If it is, but we cannot find a closed form formula, can we estimate how well the partial sum approximates the actual infinite sum? That is, any estimate of

   $$\left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n \right| ? \tag{23}$$

   In practice this is very important. For example, if we know that the difference decreases as $3/N^3$ and the problem requires 2-digit accuracy, we know it suffices to sum up the first 10 terms.

These issues are settled by the theory of power series and analytic functions.

1.2. Power series and analytic functions.

A **power series** about a point x_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \tag{24}$$

Following our previous discussion, we want to know whether this infinite sum indeed represents a function, say $F(x)$, or not. If it does, then for any c we would have

$$F(c) = \sum_{n=0}^{\infty} a_n (c - x_0)^n. \tag{25}$$

As the RHS is an infinite sum, it should be understood as

$$F(c) = \lim_{N \nearrow \infty} \sum_{n=0}^{N} a_n (c - x_0)^n. \tag{26}$$

This motivates the following.

- We say that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ **converges** at the point $x = c$ if the limit

  $$\lim_{N \nearrow \infty} \sum_{n=0}^{N} a_n (c - x_0)^n \tag{27}$$

  exists and is finite.

- If this limit does not exist, we say that the power series **diverges** at $x = c$.

Clearly, equations like

$$\sum_{n=0}^{\infty} a_n x^n = 0 \tag{28}$$

is only meaningful when the LHS converges. Recall that we would like to justify concluding $a_n = 0$ from this equation. This is fulfilled by the following theorem.

The **most important** property of power series is the following:

Theorem 2. (Radius of convergence) *For any power series $\sum a_n (x - x_0)^n$, there is a number $\rho \in [0, \infty]$ (meaning: $\rho \geqslant 0$ and can be infinity) such that*

- *the power series converges for all x such that $|x - x_0| < \rho$;*
- *the power series diverges for all x such that $|x - x_0| > \rho$.*

*This particular number ρ is called the **radius of convergence**.*

Remark 3. The number ρ is at least 0, as taking $x = x_0$ gives $\sum 0$ which is clearly converging to 0; On the other hand, when the power series is convergent for all x, we say its radius of convergence is infinity, that is $\rho = \infty$.

Remark 4. Whether the power series converges at $x = x_0 \pm \rho$ is tricky to determine. Different approaches are needed for different power series.

For those who are curious, this theorem is a consequence of the following proposition.

Proposition 5. *If the power series $\sum a_n (x - x_0)^n$ is convergent at $x = c$, then it is convergent at all x satisfying*

$$|x - x_0| < |c - x_0|. \tag{29}$$

The significance of radius of convergence is that, we can manipulate a power series almost freely inside $|x - x_0| < \rho$. In particular all our questions are satisfactorily answered.

Theorem 6. (Properties of power series inside $|x - x_0| < \rho$)

- **(Power series vanishing on an interval)** *If*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0 \tag{30}$$

for all x in some open interval, then each a_n is 0.[1]

- **(Differentiation and integration of power series)** *If the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a positive radius of convergence ρ, then if we set $f(x)$ to be the sum inside $|x - x_0| < \rho$, then f is differentiable and integrable inside this same interval. Furthermore we can perform differentiation and integration term by term.*

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1} \qquad |x - x_0| < \rho; \tag{31}$$

$$\int f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C \qquad |x - x_0| < \rho. \tag{32}$$

- **(Multiplication of power series)** *If $\sum a_n (x - x_0)^n$ has a positive radius of convergence ρ_1 and $\sum b_n (x - x_0)^n$ has a positive radius of convergence ρ_2, then termwise multiplication*

$$(a_0 + a_1(x - x_0) + \cdots)\,(b_0 + b_1 (x - x_0) + \cdots) = a_0\, b_0 + [a_0\, b_1 + a_1\, b_0]\,(x - x_0) + \cdots \tag{33}$$

is OK for all $|x - x_0| < \rho$ where $\rho := \min\{\rho_1, \rho_2\}$.

- **(Division of power series)** *If $\sum a_n (x - x_0)^n$ has a positive radius of convergence ρ_1 and $\sum b_n (x - x_0)^n$ has a positive radius of convergence ρ_2, and furthermore $\sum b_n (x - x_0)^n \neq 0$, then for $|x - x_0| < \min\{\rho_1, \rho_2\}$ one can perform the division*

$$\frac{\sum a_n (x - x_0)^n}{\sum b_n (x - x_0)^n} = \sum c_n (x - x_0)^n \tag{34}$$

by solving c_n's through

$$\sum a_n (x - x_0)^n = \Big(\sum b_n (x - x_0)^n\Big)\Big(\sum c_n (x - x_0)^n\Big). \tag{35}$$

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1. The idea of "radius of convergence" is not explicit here. But notice that $\sum a_n (x - x_0)^n = 0$ implies the convergence of the infinite sum, and consequently $|x - x_0| \leqslant \rho$.

Remark 7. There are many alternative definition of the value of infinite sums which extends the above properties outside $|x - x_0| < \rho$, that is they give meaning to $\sum a_n (x - x_0)^n$ for $|x - x_0| = \rho$ or even $|x - x_0| > \rho$. Such techniques have practical values. In particular, in many cases the "sum" (as defined by the alternative definitions) of $\sum a_n (x - x_0)^n$ for certain x with $|x - x_0| > \rho$ still approximates the value of the solution.

A few of such alternative summations are Cesaro summation, Abel summation, and Borel summation. They are parts of the theory of asymptotic analysis.

Example 8. Find a power series expansion for $f'(x)$, with

$$f(x) = (1 + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n. \tag{36}$$

Solution. We have

$$f'(x) = \sum_{n=0}^{\infty} \left[(-1)^n x^n\right]' = \sum_{n=1}^{\infty} (-1)^n n\, x^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)\, x^n. \tag{37}$$

Note the change of index range in the 2nd equality. Such "index shifting" will occur every time we try to solve an equation using power series.

Example 9. Find a power series expansion for $g(x) = \int_0^x f(t)\, \mathrm{d}t$ for

$$f(x) = (1 + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n. \tag{38}$$

Solution. Compute

$$\int f(x) = \sum_{n=0}^{\infty} (-1)^n \int x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C. \tag{39}$$

The constant C is determined through setting $x = 0$:

$$C = g(0) = 0. \tag{40}$$

Therefore

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \tag{41}$$

or if preferred,

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n. \tag{42}$$

Example 10. Compute the power series[2] for $f g$ with

$$f = e^{-x}, \quad g = (1 + x)^{-1}. \tag{43}$$

Solution. We first find the power series for f and g.

- f. We know that

  $$\left(e^{-x}\right)^{(n)} = (-1)^n e^{-x}. \tag{44}$$

  Thus

  $$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n. \tag{45}$$

  Note that the radius of convergence is ∞.

- g. We have

  $$\left(\frac{1}{1+x}\right)^{(n)} = (-1)^n n! \frac{1}{(1+x)^n}. \tag{46}$$

2. When x_0 is not specified, it is implicitly set as 0.

Therefore

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n. \tag{47}$$

The radius of convergence is 1.

Note that the radius of convergence for the power series of the product $f\,g$ is the smaller of that of f and g. So in our case here is 1.

Now we compute the product

$$\begin{aligned} f\,g &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n\right]\left[\sum_{n=0}^{\infty} (-1)^n x^n\right] \\ &= \left[1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right][1 - x + x^2 - x^3 + \cdots] \\ &= 1 - 2\,x + \frac{5}{2} x^2 + \cdots \end{aligned} \tag{48}$$

Given that this number ρ is so important, we clearly would like to have a way to compute it. We have the following theorem.

Theorem 11. (Ratio test) *If, for n large, the coefficients a_n are nonzero and satisfy*

$$\lim_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = L \qquad (0 \leqslant L \leqslant \infty) \tag{49}$$

then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is $\rho = 1/L$, with $\rho = \infty$ if $L = 0$ and $\rho = 0$ if $L = \infty$. That is, the power series converges[3] *for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$.*

Remark 12. In other words, if

$$\lim_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = L \qquad (0 \leqslant L \leqslant \infty) \tag{50}$$

then the power series coincide with a well-defined function for $|x - x_0| < \rho = 1/L$.

Remark 13. Sometimes the limit may not exist. In those cases we need to use the following more general formula:

$$\limsup_{n \longrightarrow \infty} (|a_n|)^{1/n} = L = \rho^{-1}. \tag{51}$$

Here limsup is defined as follows:

$$\limsup_{n \longrightarrow \infty} A_n = \lim_{N \longrightarrow \infty} \left[\sup_{n \geqslant N} A_n\right] \tag{52}$$

in which

$$\sup_{n \geqslant N} A_n \tag{53}$$

is the lowest upper bound of the numbers $\{A_n\}_{n \geqslant N}$, or in other words the smallest number that is larger than all A_n's with $n \geqslant N$.

For example, consider $A_n = \begin{cases} 1 + 1/n & n \text{ odd} \\ 1/n & n \text{ even} \end{cases}$. Then $\limsup_{n \longrightarrow \infty} A_n = 1$. Note that $\lim A_n$ does not exist.

Remark 14. Note that, when

$$\lim_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| \tag{54}$$

3. *In fact, absolutely converges.*

does not exist, the radius of convergence is **not** given by

$$\rho^{-1} = \limsup_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| ! \tag{55}$$

This can be easily seen from the power series

$$1 + x + (2\,x)^2 + x^3 + (2\,x)^4 + \cdots. \tag{56}$$

Example 15. Determine the convergence set for

$$\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} (x-1)^n. \tag{57}$$

Solution. We apply the ratio test. Compute

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{-(n+1)} \frac{1}{n+2}}{2^{-n} \frac{1}{n+1}} = \frac{n+1}{2\,(n+2)}. \tag{58}$$

Clearly we have

$$\lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}. \tag{59}$$

Thus the radius of convergence is 2 and the power series converges for $-1 < x < 3$ and diverges for $x < -1$ or $x > 3$.

How about -1 and 3? For these points we need to determine in an ad hoc manner.

- $x = -1$. We have

  $$\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} (x-1)^n = \sum_{n=0}^{\infty} \frac{2^{-n}}{(n+1)} (-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \tag{60}$$

  which is actually converging.[4]

- $x = 3$. We have

  $$\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} (x-1)^n = \sum_{n=0}^{\infty} \frac{2^{-n}}{(n+1)} 2^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \tag{61}$$

  which is diverging.[5]

Remark 16. As we will see soon, when solving ODEs using power series method, it is possible to know the radius of convergence **before** actually finding out the coefficients for the expansion of y. This is important – knowing ρ enables us to estimate how well the partial sum approximates the solution, based on which we can determine how many coefficients we need to compute.

We have seen that the functions that can be represented by power series have very nice properties. Let's give them a name.

Definition 17. (Analytic functions) *A function f is said to be* ***analytic at x_0*** *if, in an open interval about x_0, the function is the sum of a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ that has positive radius of convergence.*

It turns out that this power series is exactly the Taylor expansion of f:

$$f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2} f''(x_0)\,(x - x_0)^2 + \cdots \tag{62}$$

4. $\sum a_n$ is converging if a_n and a_{n+1} have different sign for each n, $\lim a_n = 0$, and $|a_n|$ is decreasing.

5. $\sum_{n=0}^{\infty} \frac{1}{n+1} \geqslant \int_1^{\infty} \frac{1}{x}$.

with general term

$$\frac{1}{n!}\, f^{(n)}(x_0)\,(x-x_0)^n. \tag{63}$$

Remark 18. Note that "f is the sum of the power series" cannot be dropped. More specifically, consider a function f, and suppose that we can expand it into Taylor series at some point x_0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n = \sum a_n\,(x-x_0)^n \tag{64}$$

Then, "The series has positive radius of convergence ρ" **does not imply** "$f = \sum a_n\,(x-x_0)^n$ for $|x-x_0| < \rho$! A typical example is

$$f(x) = 1 - e^{-\frac{1}{x^2}}. \tag{65}$$

Try computing its Taylor series at $x_0 = 0$ and see what happens.

Remark 19. As a consequence of the above remark, it would be very awkward if we have to determine analyticity of a function through the above definition. Fortunately we have other ways to do this. In particular, if f, g are analytic at x_0, so are $c_1\, f + c_2\, g$, $f\, g$, and f/g (when $g(x_0) \neq 0$) as well as their derivatives and integrals f', $\int f$. Typical analytic functions are polynomials, $\sin x$, $\cos x$, e^x, and their composites. Thus we know that $\sin\left(2\,x^2+1\right)$ is analytic everywhere, while $\frac{e^{3x+2}}{\sin x}$ is analytic at every point except those solving $\sin x = 0$.

Remark 20. In the 18th century people believe all reasonable functions (say continuous) are analytic. This clearly is the belief that fuels the research on power series method. Later in the 19th century it was discovered that this belief is wrong – most functions do not equal to any good power series. Nevertheless, the method of power series survived this change and remains a powerful tool in solving ODEs even today.

1.3. More examples.

Now we return to solving equations using power series.

Example 21. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for a general solution to

$$z'' - x^2\, z = 0. \tag{66}$$

Solution. We write

$$z(x) = a_0 + a_1\, x + \cdots = \sum_{n=0}^{\infty} a_n\, x^n. \tag{67}$$

Substituting into the equation, we have

$$\begin{aligned}
0 &= z'' - x^2\, z \\
&= \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} - \sum_{n=0}^{\infty} a_n\, x^{n+2} \\
&= \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\, x^n - \sum_{n=2}^{\infty} a_{n-2}\, x^n \\
&= 2\,a_2 + 6\,a_3\, x + \sum_{n=2}^{\infty} \left[a_{n+2}\,(n+2)\,(n+1) - a_{n-2}\right] x^n.
\end{aligned} \tag{68}$$

Thus we have

$$2\,a_2 = 0 \tag{69}$$

$$6\,a_3 = 0 \tag{70}$$

$$a_{n+2}\,(n+2)\,(n+1) = a_{n-2} \Longrightarrow a_{n+2} = \frac{a_{n-2}}{(n+2)\,(n+1)}. \tag{71}$$

We conclude:

$$\begin{aligned} a_2 &= 0 & (72)\\ a_3 &= 0 & (73)\\ a_4 &= \frac{a_0}{12} & (74)\\ a_5 &= \frac{a_1}{20} & (75)\end{aligned}$$

As we only need 4 nonzero terms, we stop here. The solution is

$$z(x) = a_0 + a_1\, x + \frac{a_0}{12}\, x^4 + \frac{a_1}{20}\, x^5 + \cdots. \tag{76}$$

Example 22. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients.

$$y'' - x\, y' + 4\, y = 0. \tag{77}$$

Solution. We write

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n. \tag{78}$$

Substituting into the equation, we have

$$\begin{aligned} 0 &= y'' - x\, y' + 4\, y\\ &= \sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} - x \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + \sum_{n=0}^{\infty} 4\, a_n\, x^n\\ &= \sum_{n=0}^{\infty} a_{n+2}\, (n+2)\, (n+1)\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n + \sum_{n=0}^{\infty} 4\, a_n\, x^n\\ &= (2\, a_2 + 4\, a_0) + \sum_{n=1}^{\infty} [(n+2)\, (n+1)\, a_{n+2} - (n-4)\, a_n]\, x^n. & (79)\end{aligned}$$

This gives

$$\begin{aligned} 2\, a_2 + 4\, a_0 &= 0 & (80)\\ (n+2)\, (n+1)\, a_{n+2} - (n-4)\, a_n &= 0. & (81)\end{aligned}$$

Therefore

$$\begin{aligned} a_2 &= -2\, a_0, & (82)\\ a_{n+2} &= \frac{n-4}{(n+2)\, (n+1)}\, a_n. & (83)\end{aligned}$$

It is clear that we should discuss $n = 2\, k$ and $n = 2\, k - 1$ separately.

For even n, we have

$$a_2 = -2\, a_0,\ \ a_4 = -\frac{1}{6}\, a_2 = \frac{1}{3}\, a_0,\ \ a_6 = 0,\ \ a_8 = 0,\ \ a_{10} = 0, \ldots \tag{84}$$

For odd n, we have

$$a_{2k+1} = \frac{2\, k - 5}{(2\, k+1)\, (2\, k)}\, a_{2k-1} = \frac{(2\, k-5)\, (2\, k-7)}{(2\, k+1) \cdots (2\, k-2)}\, a_{2k-3} = \cdots = \frac{(2\, k-5) \cdots (-3)}{(2\, k+1)!}\, a_1. \tag{85}$$

Summarizing, we have

$$y(x) = a_0 \left[1 - 2\, x^2 + \frac{1}{3}\, x^4\right] + a_1 \left[x + \sum_{k=1}^{\infty} \frac{(2\, k-5) \cdots (-3)}{(2\, k+1)!}\, x^{2k+1}\right]. \tag{86}$$

Example 23. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem.

$$w'' + 3\, x\, w' - w = 0, \qquad w(0) = 2, \quad w'(0) = 0. \tag{87}$$

Solution. We write

$$w(x)=\sum_{n=0}^{\infty} a_n x^n. \tag{88}$$

Substituting into the equation, we obtain

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=1}^{\infty} 3 n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + 3 n a_n - a_n] x^n. \end{aligned} \tag{89}$$

Therefore

$$2a_2 - a_0 = 0 \tag{90}$$

$$(n+2)(n+1) a_{n+2} + (3n-1) a_n = 0 \tag{91}$$

which leads to

$$a_2 = \frac{1}{2} a_0 \tag{92}$$

$$a_{n+2} = \frac{1-3n}{(n+2)(n+1)} a_n. \tag{93}$$

On the other hand, the initial values give

$$2 = w(0) = a_0, \quad 0 = w'(0) = a_1. \tag{94}$$

Therefore we can compute successively

$$a_2 = \frac{1}{2} a_0 = 1, \tag{95}$$

$$a_3 = \frac{-2}{6} a_1 = 0, \tag{96}$$

$$a_4 = \frac{-5}{12} a_2 = -\frac{5}{12}, \tag{97}$$

$$a_5 = \frac{-8}{20} a_3 = 0, \tag{98}$$

$$a_6 = \frac{-11}{30} a_4 = \frac{11}{72}. \tag{99}$$

We stop here as only four nonzero terms are required. Finally the answer is

$$w(x) = 2 + x^2 - \frac{5}{12} x^4 + \frac{11}{72} x^6 + \cdots. \tag{100}$$

Remark 24. Note that, there is no way to know a priori how many terms we need to compute before getting four nonzero terms.

1.4. Ordinary and singular points.

Not all linear ODEs are amenable to the above naïve approach

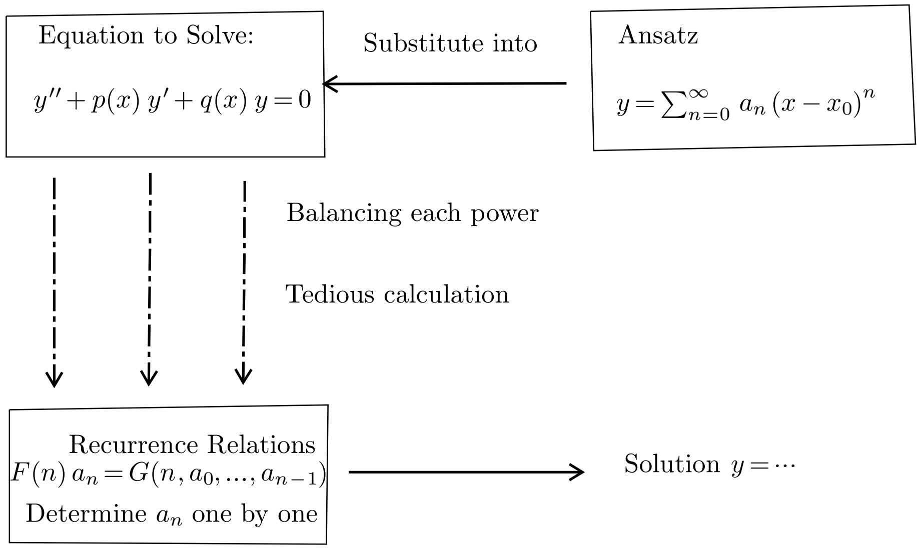

Figure 1. Naïve Power Series Method

: Write expansion $\longrightarrow$ Substitute into equation $\longrightarrow$ Determine coefficients.

Example 25. Consider

$$x^2\, y'' + 3\, y' - x\, y = 0. \tag{101}$$

Solution. Write

$$y = \sum_{n=0}^{\infty} a_n\, x^n. \tag{102}$$

Substituting into the equation, we obtain

$$\begin{aligned} 0 &= x^2 \sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} + 3 \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} - \sum_{n=0}^{\infty} a_n\, x^{n+1} \\ &= \sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^n + \sum_{n=0}^{\infty} 3\, (n+1)\, a_{n+1}\, x^n - \sum_{n=1}^{\infty} a_{n-1}\, x^n \\ &= 3\, a_1 + (6\, a_2 - a_0)\, x + \sum_{n=2}^{\infty} [a_n\, n\, (n-1) + 3\, (n+1)\, a_{n+1} - a_{n-1}]\, x^n. \end{aligned} \tag{103}$$

This leads to

$$3\, a_1 = 0 \tag{104}$$

$$6\, a_2 - a_0 = 0 \tag{105}$$

$$3\, (n+1)\, a_{n+1} + n\, (n-1)\, a_n - a_{n-1} = 0, \qquad n \geqslant 2 \tag{106}$$

which leads to

$$a_1 = 0, \quad a_2 = \frac{a_0}{6}, \quad a_{n+1} = \frac{a_{n-1} - n\, (n-1)\, a_n}{3\, (n+1)}. \tag{107}$$

Clearly, all coefficients are determined once a_0 is given. In other words, the power series solution we obtain has only one arbitrary constant. As the equation is of second order, this means not all solutions can be obtained through expansion into power series.

Example 26. Let's consider the Euler equation

$$x^2\, y'' + y = 0. \tag{108}$$

Solution. Assume the ansatz

$$y = \sum_{n=0}^{\infty} a_n\, x^n. \tag{109}$$

Substitute into the equation:

$$x^2 \sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = 0. \tag{110}$$

This is simplified to

$$0 = \sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1\, x + \sum_{n=2}^{\infty} [n\, (n-1) + 1]\, a_n\, x^n. \tag{111}$$

The recurrence relations are

$$a_0 = 0 \tag{112}$$

$$a_1 = 0 \tag{113}$$

$$[n\, (n-1) + 1]\, a_n = 0 \quad n \geqslant 2 \tag{114}$$

Clearly we have $a_n = 0$ for all n. So the power series method only yield the trivial solution $y = 0$. In other words, all the above calculation is a pure waste of time.

Remark 27. We know how to solve the above equation: Guess $y = x^r$ to obtain

$$r\, (r-1) + 1 = 0 \Longrightarrow r_{1,2} = \frac{1 \pm \sqrt{3}\, i}{2} \tag{115}$$

which means the general solution is

$$y = C_1\, x^{1/2} \cos\left(\sqrt{3}\, \ln|x|\right) + C_2\, x^{1/2} \sin\left(\sqrt{3}\, \ln|x|\right). \tag{116}$$

No wonder we cannot obtain any solution using power series!

Checking the difference between this example and the previous ones, we reach the following definition.

Definition 28. (Ordinary and singular points) *Consider the linear differential equation in the standard form*

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{117}$$

A point x_0 is called an ***ordinary point*** *if both p, q are analytic at x_0. If x_0 is not an ordinary point, it is called a* ***singular point*** *of the equation.*

Example 29. Find the singular points of the equation we just discussed.

$$x^2\, y'' + 3\, y' - x\, y = 0. \tag{118}$$

Solution. First write it into the standard form

$$y'' + \frac{3}{x^2}\, y' - \frac{1}{x}\, y = 0. \tag{119}$$

We have

$$p(x) = \frac{3}{x^2}, \quad q(x) = -\frac{1}{x}. \tag{120}$$

As $3, 1, x^2$ and x are all analytic everywhere, their ratios are analytic at all points except those making the denominator vanish.

Therefore the only singular point is $x = 0$.

Remark 30. When x_0 is an ordinary point, we expect no trouble finding the general solution by setting

$$y(x)=\sum_{n=0}^{\infty} a_n\,(x-x_0)^n. \tag{121}$$

In other words, when x_0 is an ordinary point, all solutions can be expanded into the above form.

When x_0 is singular, however, in general not all solutions can be represented in the above form, as we have seen in the example.

Remark 31. It turns out that, when x_0 is singular, there are still two cases.

- If $p(x)\,(x-x_0)$, $q(x)\,(x-x_0)^2$ are analytic, then the solution can be solved via a generalized version of the series method; Such x_0 is called **regular singular**.
- In all other cases, x_0 is called **irregular singular** and there is no universally good way of solving the equation.

2. Equations with analytic coefficients.

The following theorem summarizes the good properties of ordinary points.

Theorem 32. (Existence of analytic solutions) *Suppose x_0 is an ordinary point for equation*

$$y''(x)+p(x)\,y'(x)+q(x)\,y(x)=0, \tag{122}$$

then it has two linearly independent analytic solutions of the form

$$y(x)=\sum_{n=0}^{\infty} a_n\,(x-x_0)^n. \tag{123}$$

Moreover, the radius of convergence of any power series solution of the form given above is at least as large as the distance from x_0 to the nearest singular point (real or complex-valued).

Remark 33. It is important, when determining the radius of convergence, to remember count in complex singular points.

Remark 34. Recall that, for many problems, we could not write down an explicit formula for the coefficients a_n, but can only compute a few terms. For example, when considering

$$w''+3\,x\,w'-w=0, \qquad w(0)=2, \quad w'(0)=0. \tag{124}$$

we obtain

$$w(x)=2+x^2-\frac{5}{12}\,x^4+\frac{11}{72}\,x^6+\cdots. \tag{125}$$

With the help of this theorem, we can estiamte how good the first few terms approximates $w(x)$, that is we can estimate the size of

$$w(x)-\left(2+x^2-\frac{5}{12}\,x^4+\frac{11}{72}\,x^6\right). \tag{126}$$

Example 35. Find a minimum value for the radius of convergence of a power series solution about x_0.

$$(x+1)\,y''-3\,x\,y'+2\,y=0, \qquad x_0=1. \tag{127}$$

Solution. Write the equation to standard form

$$y''-\frac{3\,x}{x+1}\,y'+\frac{2}{x+1}\,y=0. \tag{128}$$

The only singular point is $x=-1$. Thus the minimum radius of convergence is the distance between $x_0=1$ and -1, which is 2.

Example 36. Find a minimum value for the radius of convergence of a power series solution about x_0.

$$\left(1+x+x^2\right) y'' - 3\,y = 0; \qquad x_0 = 1. \tag{129}$$

Solution. Write the equation to standard form

$$y'' - \frac{3}{1+x+x^2}\,y = 0. \tag{130}$$

The singular points are roots of $1+x+x^2$, which are

$$x_{1,2} = \frac{-1 \pm \sqrt{3}\,i}{2}. \tag{131}$$

To find out the closest singular point to x_0, we compute

$$|x_1 - x_0| = \left| -\frac{3}{2} + \frac{\sqrt{3}}{2}\,i \right| = \sqrt{3}; \qquad |x_2 - x_0| = \sqrt{3}. \tag{132}$$

So both are $\sqrt{3}$ away from x_0. As a consequence, the minimum radius of convergence is $\sqrt{3}$.

Example 37. Find at least the first four nonzero terms in a power series expansion about x_0 for a general solution to the given differential equation with the given value of x_0.

$$\left(x^2 - 2\,x\right) y'' + 2\,y = 0, \qquad x_0 = 1. \tag{133}$$

Solution. The best way to do this is to first shift x_0 to 0. To do this, let $t = x - 1$. Then $t_0 = x_0 - 1 = 0$, $x^2 - 2\,x = t^2 - 1$, and the equation becomes

$$\left(t^2 - 1\right) y'' + 2\,y = 0 \tag{134}$$

and we would like to expand at $t_0 = 0$.

Substituting

$$y = \sum_{n=0}^{\infty} a_n\, t^n \tag{135}$$

into the equation, we have

$$\begin{aligned} 0 &= \left(t^2 - 1\right) \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, t^{n-2} + 2 \sum_{n=0}^{\infty} a_n\, t^n \\ &= \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, t^n - \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, t^{n-2} + \sum_{n=0}^{\infty} 2\,a_n\, t^n \\ &= \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, t^n - \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\, t^n + \sum_{n=0}^{\infty} 2\,a_n\, t^n \\ &= (-2\,a_2 + 2\,a_0) + (-6\,a_3 + 2\,a_1)\; t + \sum_{n=2}^{\infty} \left[a_n\, n\,(n-1) - a_{n+2}\,(n+2)\,(n+1) + 2\,a_n\right] t^n. \end{aligned} \tag{136}$$

This leads to

$$-2\,a_2 + 2\,a_0 = 0 \tag{137}$$

$$-6\,a_3 + 2\,a_1 = 0 \tag{138}$$

$$a_n\, n\,(n-1) - a_{n+2}\,(n+2)\,(n+1) + 2\,a_n = 0. \tag{139}$$

As only four terms are needed, all we need to settle is

$$a_2 = a_0, \quad a_3 = \frac{1}{3}\,a_1. \tag{140}$$

Thus

$$y(t) = a_0 + a_1\, t + a_0\, t^2 + \frac{1}{3}\,a_1\, t^3 + \cdots \tag{141}$$

Back to the x variable, we have

$$y(x) = a_0 \left[1 + (x-1)^2 + \cdots \right] + a_1 \left[(x-1) + \frac{1}{3} (x-1)^3 + \cdots \right]. \tag{142}$$

Example 38. Find at least the first four nonzero terms in a power series expansion of the solution to the given initial value problem.

$$y'' - (\sin x)\, y = 0, \qquad y(\pi) = 1, \quad y'(\pi) = 0. \tag{143}$$

Solution. As the initial values are given at π, the expansion should be about $x_0 = \pi$. First introduce new variable $t = x - \pi$. Then $x = t + \pi$ and the equation becomes

$$y'' + (\sin t)\, y = 0, \qquad y(0) = 1, \quad y'(0) = 0. \tag{144}$$

Now write

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \tag{145}$$

and substitute into the equation, recalling

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}, \tag{146}$$

we reach

$$\begin{aligned} 0 &= y'' + (\sin t)\, y \\ &= \sum_{n=2}^{\infty} a_n n (n-1) t^{n-2} + \left(t - \frac{1}{3!} t^3 + \cdots \right) (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots) \\ &= 2 a_2 + 6 a_3 t + 12 a_4 t^2 + 20 a_5 t^3 + \cdots \\ &\quad + a_0 t + a_1 t^2 + \left(a_2 - \frac{a_0}{6} \right) t^3 + \cdots \\ &= 2 a_2 + (6 a_3 + a_0) t + (12 a_4 + a_1) t^2 + \left(20 a_5 + a_2 - \frac{a_0}{6} \right) t^3 + \cdots \end{aligned} \tag{147}$$

This gives

$$\begin{aligned} 2 a_2 &= 0 & (148) \\ 6 a_3 + a_0 &= 0 & (149) \\ 12 a_4 + a_1 &= 0 & (150) \\ 20 a_5 + a_2 - \frac{a_0}{6} &= 0 & (151) \end{aligned}$$

Applying the initial values we have

$$a_0 = 1, \quad a_1 = 0. \tag{152}$$

Thus the above leads to

$$a_0 = 1,\ a_1 = 0,\ a_2 = 0,\ a_3 = -\frac{1}{6},\ a_4 = 0,\ a_5 = \frac{1}{120}. \tag{153}$$

Only three of them are non-zero. We have to return to the equation and expand to higher orders.

$$\begin{aligned} 0 &= y'' + (\sin t)\, y \\ &= \sum_{n=2}^{\infty} a_n n (n-1) t^{n-2} + \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \cdots \right) (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \cdots) \\ &= 2 a_2 + 6 a_3 t + 12 a_4 t^2 + 20 a_5 t^3 + 30 a_6 t^4 + \cdots \\ &\quad a_0 t + a_1 t^2 + \left(a_2 - \frac{a_0}{6} \right) t^3 + \left(a_3 - \frac{a_1}{6} \right) t^4 + \cdots \end{aligned} \tag{154}$$

which gives

$$30 a_6 + a_3 - \frac{a_1}{6} = 0 \tag{155}$$

from balancing the t^4 term. As $a_1 = 0, a_3 = -1/6$, we have

$$a_6 = \frac{1}{180}. \tag{156}$$

As $a_6 \neq 0$, we already have 4 nonzero terms and do not need to go to higher orders. The solution in t variable is

$$y(t) = 1 - \frac{1}{6} t^3 + \frac{1}{120} t^5 + \frac{1}{180} t^6 + \cdots \tag{157}$$

Returning to the x-variable, we have

$$y(x) = 1 - \frac{1}{6} (x - \pi)^3 + \frac{1}{120} (x - \pi)^5 + \frac{1}{180} (x - \pi)^6 + \cdots \tag{158}$$

There is no difficulty extending the power series method to non-homogeneous problems.

Remark 39. When $p(x)$ or $q(x)$ has a power series expansion involving more than a few terms (as in the above example, $q(x) = -\sin x$ and its expansion at 1 involves infinitely many terms), in general it is not easy to write down the general formula for the coefficients of x^n in the product. Thus we encounter the following question: When computing

$$p(x)\, y' \text{ or } q(x)\, y, \tag{159}$$

how many terms should we keep in the expansion of p, q, y?

Let's revisit the situation in the last example. We need to get the first few terms of

$$\left(t - \frac{1}{3!} t^3 + \cdots \right) \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \right). \tag{160}$$

and later we see that expanding to t^3 is not enough.

Unfortunately there is no general rule of deciding how many terms we should keep. Nevertheless, there are simple rules that may make the "trial and error" procedure more efficient.

1. Look at the whole equation.

$$\sum_{n=2}^{\infty} a_n\, n\, (n-1)\, t^{n-2} + \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \cdots \right) \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \cdots \right) \tag{161}$$

   We know that the constant term would give us a_2, the t term would give us a_3, the t^2 term would give us a_4, and so on.

   As we need to get the "first four nonzero terms" of y, we have to at least compute a_0, a_1, a_2, a_3. Thus the expansion has to be at least up to t;

   Now study the problem a bit more carefully, we see that the initial condition $y'(\pi) = 0$ leads to $a_1 = 0$. As a consequence, to get four nonzero terms, we have to compute at least up to a_4. As a consequence, we need to expand at least up to the term t^2.

2. Consider two power series

$$\sum_{n=0}^{\infty} p_n\, x^n, \qquad \sum_{n=0}^{\infty} q_n\, x^n. \tag{162}$$

   Suppose we need to compute their product:

$$\left(p_0 + p_1 x + p_2 x^2 + \cdots \right) \left(q_0 + q_1 x + q_2 x^2 + \cdots \right) \tag{163}$$

   up to at least the power x^m. Then how many terms in each expansion should we keep? The answer is exactly m. For example, if we want to get the correct coefficient for x^3 in the above product, then writing the product as

$$\left(p_0 + p_1 x + p_2 x^2 + \cdots \right) \left(q_0 + q_1 x + q_2 x^2 + \cdots \right) \tag{164}$$

is not enough! We need to write each power series to the x^3 term:

$$\left(p_0 + p_1\,x + p_2\,x^2 + p_3\,x^3 + \cdots\right)\left(q_0 + q_1\,x + q_2\,x^2 + q_3\,x^3 + \cdots\right) \tag{165}$$

and the coefficients for $1, x, x^2, x^3$ are exactly the coefficients for $1, x, x^2, x^3$ in the product

$$\left(p_0 + p_1\,x + p_2\,x^2 + p_3\,x^3\right)\left(q_0 + q_1\,x + q_2\,x^2 + q_3\,x^3\right). \tag{166}$$

3. Suppose writing each power series up to x^3 is not enough, and we try to compute further the coefficient of x^4. That is the coefficient of x^4 in the product

$$\left(p_0 + p_1\,x + p_2\,x^2 + p_3\,x^3 + p_4\,x^4\right)\left(q_0 + q_1\,x + q_2\,x^2 + q_3\,x^3 + q_4\,x^4\right). \tag{167}$$

The good news is that, we do not need to re-compute the $1, x, x^2, x^3$ term. All we need to do is to focus on the terms in the two sums that will give us x^4: $p_0 \cdot q_4\,x^4$, $p_1\,x \cdot q_3\,x^3$, and so on.

Applying the above understanding to the last example. We see that the most optimal guess is that it's sufficient to compute

$$(t + \cdots)\left(a_0 + a_1\,t + a_2\,t^2 + \cdots\right) \tag{168}$$

that is expanding each up to t^2. Of course after some algebra we find out that this is not enough, as $a_2 = a_4 = 0$. So expanding up to t^2 we only obtain two nonzero coefficients: a_0, a_3. To get four, we need to expand at least two more terms, that is up to t^4. It turns out that this is enough.

Example 40. Find at least the first four nonzero terms in a power series expansion about $x = 0$ of a general solution to the given differential equation.

$$y'' - x\,y' + 2\,y = \cos x. \tag{169}$$

Solution. Write

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^n \tag{170}$$

and substitute into the equation, recalling

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n)!}\,x^{2n}, \tag{171}$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n)!}\,x^{2n} = \cos x &= y'' - x\,y' + 2\,y \\
&= \sum_{n=2}^{\infty} a_n\,n\,(n-1)\,x^{n-2} - \sum_{n=1}^{\infty} n\,a_n\,x^n + \sum_{n=0}^{\infty} 2\,a_n\,x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,x^n - \sum_{n=1}^{\infty} n\,a_n\,x^n + \sum_{n=0}^{\infty} 2\,a_n\,x^n \\
&= (2\,a_2 + 2\,a_0) + \sum_{n=1}^{\infty} \left[a_{n+2}\,(n+2)\,(n+1) - n\,a_n + 2\,a_n\right] x^n.
\end{aligned} \tag{172}$$

The first few terms of balance is then:

$$x^0: \quad 1 = 2\,a_2 + 2\,a_0 \tag{173}$$

$$x^1: \quad 0 = 6\,a_3 + a_1 \tag{174}$$

$$x^2: \quad \frac{1}{2} = 12\,a_4 \tag{175}$$

From these we have

$$a_2 = \frac{1}{2} - a_0, \qquad a_3 = -\frac{1}{6}\,a_1. \tag{176}$$

So the solution is

$$\begin{aligned} y(x) &= a_0 + a_1\,x + \left(\frac{1}{2} - a_0\right) x^2 - \frac{1}{6}\,a_1\,x^3 + \cdots \\ &= \left(\frac{1}{2}\,x^2 + \cdots\right) + a_0\,(1 - x^2 + \cdots) + a_1\left(x - \frac{1}{6}\,x^3 + \cdots\right). \end{aligned} \tag{177}$$

Remark 41. From the above we can conlude that the expansions of y_p, y_1, y_2 are

$$y_p(x) = \frac{1}{2}\,x^2 + \cdots \tag{178}$$

$$y_1(x) = 1 - x^2 + \cdots \tag{179}$$

$$y_2(x) = x - \frac{1}{6}\,x^3 + \cdots \tag{180}$$

(Don't forget that the choices of y_p, y_1, y_2 are not unique!)

Remark 42. The power series method is equivalent to the following idea of solving linear ODEs. Consider a second order ODE

$$y'' + p(x)\,y' + q(x)\,y = 0, \qquad y(x_0) = a_0, \quad y'(x_0) = a_1. \tag{181}$$

Then setting $x = x_0$ in the equation, we get

$$y''(x_0) = -\,p(x_0)\,y'(x_0) - q(x_0)\,y(x_0) = -\,p(x_0)\,a_1 - q(x_0)\,a_0. \tag{182}$$

To determine y''', we differentiate the equation:

$$y''' + p\,y'' + (p' + q)\,y' + q'\,y = 0. \tag{183}$$

Now setting $x = x_0$ we obtain $y'''(x_0)$. Differentiating once more we get $y^{(4)}(x_0)$ at so on. After obtaining all derivatives of y at x_0, we can obtain y through Taylor expansion:

$$y = y(x_0) + y'(x_0)\,(x - x_0) + \cdots. \tag{184}$$

This approach has theoretical advantage and can be used to prove the existence and uniqueness of solutions to ODEs when everything is analytic[6]. On the other hand, simply writing $y = a_0 + a_1\,(x - x_0) + \cdots$ and substitute into the equation is in practice much easier to do.

3. Dealing with Singular Points.

In the above we have seen that, using power series about an ordinary point, we can easily obtain the power series representations for the general solution. On the other hand, if we expand about a singular point, not all solutions can be obtained through power series.

Remark 43. (Why bother?) Why should we try to solve the equation around singular points? If we want to know what happens around a singular point, why couldn't we just pick a regular point nearby, find the solution, and then study the solution near the singular point? Afterall, this solution around the regular point should converge for points near the singular point.

Unfortunately, the above strategy will not give us what we want. Because as x approaches the singular point, the convergence of the power series (obtained by solving the equation at a nearby regular point) becomes worse and worse. And letting x approach the singular point would not give us useful information.

3.1. Motivation.

A typical equation with singular point is the Cauchy-Euler equation

$$a\,x^2\,y''(x) + b\,x\,y'(x) + c\,y(x) = 0, \qquad x > 0 \tag{185}$$

where a, b, c are constants. We have studied this equation before, the conclusions are

- To find the solutions, we need to consider the associated characteristic, or indicial, equation

$$a\,r^2 + (b - a)\,r + c = 0 \tag{186}$$

6. So it's a special case of the so-called Caucky-Kowalevskaya theorem in PDE.

which can be obtained by substituting $y = x^r$ into the equation.

- There are three cases.

  1. The characteristic equation has two distinct real roots r_1, r_2. Then the general solution is given by
  $$y = c_1 x^{r_1} + c_2 x^{r_2}. \tag{187}$$
  2. The characteristic equation has one double root $r = r_0$. Then
  $$y = c_1 x^{r_0} + c_2 x^{r_0} \ln x. \tag{188}$$
  3. The characteristic equation has two complex roots $\alpha \pm i\beta$. Then
  $$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x). \tag{189}$$

From these we clearly see that, unless r_1, r_2 are both non-negative integers, there is no way that we can solve the equation by setting
$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{190}$$
since none of the solutions would be analytic around 0.

We also observe that in case 2, the second linearly independent solution, $x^{r_0} \ln x$, can be obtained formally from the first one by differentiating with respect to r:
$$x^{r_0} \ln x = (\ln x)\, e^{r_0 \ln x} = \frac{\partial}{\partial r} \left[e^{r \ln x} \right] |_{r = r_0} = \frac{\partial}{\partial r} (x^r) \,|_{r = r_0}. \tag{191}$$
To understand why this would work, recall that
$$a\,(x^r)'' + b\,(x^r)' + c\,x^r = \left[a\,r^2 + (b - a)\,r + c \right] x^r \tag{192}$$
When the characteristic equation has a double root r_0, the above becomes
$$a\,(x^r)'' + b\,(x^r)' + c\,x^r = a\,(r - r_0)^2\, x^r. \tag{193}$$
Now differentiating both sides with respect to r, we have
$$a\,(x^r \ln x)'' + b\,(x^r \ln x)' + c\,(x^r \ln x) = 2\,a\,(r - r_0)\,x^r + a\,(r - r_0)^2\, x^r \ln x. \tag{194}$$
Setting $r = r_0$, we see that $x^{r_0} \ln x$ is indeed a solution.

Finally note that, case 3 and case 1 can be unified if we consider complex variables. Since
$$x^{\alpha} \cos(\beta \ln x) \text{ and } x^{\alpha} \sin(\beta \ln x). \tag{195}$$
are simply the real and imaginary parts of $x^{\alpha + i\beta}$.

The above observations play important roles in the so-called Method of Frobenius for solving problems at singular points using power series.

3.2. The Method of Frobenius.

First we check what goes wrong when we try to solve the Cauchy-Euler equations. There are two situations.

- The first and the 3rd cases. Unless both r_1, r_2 are integers, the powers
$$x^{r_1}, x^{r_2} \tag{196}$$
cannot be represented by a power series. However, we notice that they can be represented by
$$x^r \sum a_n x^n \tag{197}$$
where r is allowed to be any complex number. When r_1, r_2 are complex, $r_{1,2} = \alpha \pm i\,\beta$, we can replace x^{r_1}, x^{r_2} by
$$x^{\alpha} \cos(\beta \ln x), \quad x^{\alpha} \sin(\beta \ln x). \tag{198}$$

- The 2nd case. Here a $\ln x$ term is involved which prevents representation even in the above modified form. However, we notice that $x^r \ln x$ can be obtained through differentiating the first solution x^r. Also note that, the first solution x^r is covered by the ansatz

$$x^r \sum a_n x^n. \tag{199}$$

The method of Frobenius is a modification to the power series method guided by the above observation. This method is effective at regular singular points. The basic idea is to look for solutions of the form

$$(x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{200}$$

Consider the equation

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{201}$$

Let x_0 be a regular singular point. That is

$$p(x)\,(x - x_0) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad q(x)\,(x - x_0)^2 = \sum_{n=0}^{\infty} q_n (x - x_0)^n \tag{202}$$

with certain radii of convergence.

To make the following discussion easier to read, we assume $x_0 = 0$.

Substitute the expansion

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \tag{203}$$

into the equation we get

$$\left(x^r \sum_{n=0}^{\infty} a_n x^n \right)'' + p(x) \left(x^r \sum_{n=0}^{\infty} a_n x^n \right)' + q(x)\, x^r \sum_{n=0}^{\infty} a_n x^n = 0. \tag{204}$$

Now compute

$$\begin{aligned} \left(x^r \sum_{n=0}^{\infty} a_n x^n \right)'' &= \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right)'' \\ &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}. \end{aligned} \tag{205}$$

$$\begin{aligned} p(x) \left(x^r \sum_{n=0}^{\infty} a_n x^n \right)' &= p(x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right)' \\ &= p(x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) \\ &= (p(x)\, x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-2} \right) \\ &= \left(\sum_{n=0}^{\infty} p_n x^n \right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-2} \right) \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} p_{n-m} (m+r)\, a_m \right\} x^{n+r-2}. \end{aligned} \tag{206}$$

$$\begin{aligned} q(x)\, x^r \sum_{n=0}^{\infty} a_n x^n &= x^{r-2} \left(\sum_{n=0}^{\infty} q_n x^n \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} q_{n-m}\, a_m \right] x^{n+r-2}. \end{aligned} \tag{207}$$

Now the equation becomes

$$\sum_{n=0}^{\infty} \left\{ (n+r)(n+r-1)\ a_n + \sum_{m=0}^{n} \left[(m+r)\, p_{n-m} + q_{n-m} \right] a_m \right\} x^{n+r-2} = 0. \tag{208}$$

Or equivalently

$$\sum_{n=0}^{\infty} \left\{ \left[(n+r)(n+r-1) + (n+r)\, p_0 + q_0 \right] a_n + \sum_{m=0}^{n-1} \left[(m+r)\, p_{n-m} + q_{n-m} \right] a_m \right\} x^{n+r-2} = 0. \tag{209}$$

This leads to the following equations:

$$(n=0): \qquad [r\,(r-1) + p_0\, r + q_0]\, a_0 \;=\; 0, \tag{210}$$

$$(n \geqslant 1): \qquad \left[(n+r)(n+r-1) + (n+r)\, p_0 + q_0 \right] a_n + \sum_{m=0}^{n-1} \left[(m+r)\, p_{n-m} + q_{n-m} \right] a_m \;=\; 0. \tag{211}$$

The $n = 0$ equation is singled out because if we require $a_0 \neq 0$ (which is natural as when $a_0 = 0$, we have $y = x^{r+1} \sum_{m=0}^{\infty} b_m\, x^m$ where $b_m = a_{m+1}$.), then it becomes a condition on r:

$$r\,(r-1) + p_0\, r + q_0 = 0. \tag{212}$$

This is called the **indicial** equation and will provide us with two roots r_1, r_2 (Some complicated situation may arise, we will discuss them later). These two roots are called **exponents** of the regular singular point $x = 0$. After deciding r, the $n \geqslant 1$ relations provide us with a way to determine a_n one by one.

Remark 44. Recall that $p(x)\, x = \sum_{n=0}^{\infty} p_n\, x^n$, $q(x)\, x^2 = \sum_{n=0}^{\infty} q_n\, x^n$. Now consider the case when $x = 0$ is regular. In this case we have $p_0 = q_0 = 0$. And the indicial equation gives $r_1 = 0$ and $r_2 = 0$. So we expect two linearly independent solutions $y_1 = a_0 + \cdots$ and $y_2 = a_1\, x + \cdots$. This is indeed what we obtained when solving equations at regular points!

It turns out that there are three cases: $r_1 \neq r_2$ with $r_1 - r_2$ not an integer; $r_1 = r_2$; $r_1 - r_2$ is an integer. Before we discuss these cases in a bit more detail, let's state the following theorem which summarizes the method of Frobenius in its full glory.

Theorem 45. *Consider the equation*

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{213}$$

at an regular singular point x_0. Let ρ be no bigger than the radius of convergence of either $(x - x_0)\, p$ or $(x - x_0)^2\, q$. Let r_1, r_2 solve the indicial equation

$$r\,(r-1) + p_0\, r + q_0 = 0. \tag{214}$$

Then

1. *If $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, then the two linearly independent solutions are given by*

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n\, (x - x_0)^n, \quad y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} \bar{a}_n\, (x - x_0)^n. \tag{215}$$

   The coefficients a_n and $\bar{a}_n$ should be determined through the recursive relation

$$\left[(n+r)(n+r-1) + (n+r)\, p_0 + q_0 \right] a_n + \sum_{m=0}^{n-1} \left[(m+r)\, p_{n-m} + q_{n-m} \right] a_m = 0. \tag{216}$$

2. *If $r_1 = r_2$, then y_1 is given by the same formula as above, and y_2 is of the form*

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{r_1} \sum_{n=1}^{\infty} d_n\, (x - x_0)^n. \tag{217}$$

3. *If $r_1 - r_2$ is an integer, then take r_1 to be the larger root (More precisely, when r_1, r_2 are both complex, take r_1 to be the one with larger real part, that is $\mathrm{Re}(r_1) \geqslant \mathrm{Re}(r_2)$). Then y_1 is still the same, while*

$$y_2(x) = c\, y_1(x) \ln|x - x_0| + |x - x_0|^{r_2} \sum_{n=0}^{\infty} e_n\, (x - x_0)^n. \tag{218}$$

Note that c may be 0.

All the solutions constructed above converge at least for $0 < |x - x_0| < \rho$ (Remember that x_0 is a singular point, so we cannot expect convergence there).

Remark 46. Note that, although ρ is given by radii of convergence of $(x - x_0)\, p$ and $(x - x_0)^2\, q$, in practice, it is the same as the distance from x_0 to the nearest singular point of p and q – no $(x - x_0)$ factor needed.

Now we discuss these cases in more detail.

Case 1: $r_1 - r_2$ is not an integer.

This case is the simplest. We work through an example.

Example 47. Solve

$$x^2\, y'' + x \left(x - \frac{1}{2} \right) y' + \frac{1}{2}\, y = 0 \tag{219}$$

at $x_0 = 0$.

Solution. We first write it into the standard form

$$y'' + \frac{(x - 1/2)}{x}\, y' + \frac{1}{2\, x^2}\, y = 0. \tag{220}$$

Thus $p(x) = \frac{x - 1/2}{x}$ and $q(x) = \frac{1}{2\, x^2}$. It is clear that $x\, p(x)$ and $x^2\, q(x)$ are analytic so 0 is a regular singular point, and the method of Frobenius applies.

Now we write

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+r}. \tag{221}$$

Substitute into the equation, we have

$$\left(\sum_{n=0}^{\infty} a_n\, x^{n+r} \right)'' + \frac{x - 1/2}{x} \left(\sum_{n=0}^{\infty} a_n\, x^{n+r} \right)' + \frac{1}{2\, x^2} \sum_{n=0}^{\infty} a_n\, x^{n+r} = 0. \tag{222}$$

As p and q are particularly simple, we write the equation as

$$\left(\sum_{n=0}^{\infty} a_n\, x^{n+r} \right)'' + \left(\sum_{n=0}^{\infty} a_n\, x^{n+r} \right)' - \frac{1}{2\, x} \left(\sum_{n=0}^{\infty} a_n\, x^{n+r} \right)' + \frac{1}{2\, x^2} \sum_{n=0}^{\infty} a_n\, x^{n+r} = 0. \tag{223}$$

Carrying out the differentiation, we reach

$$\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n\, x^{n+r-2} + \sum_{n=0}^{\infty} (n + r)\, a_n\, x^{n+r-1} - \frac{1}{2} \sum_{n=0}^{\infty} (n + r)\, a_n\, x^{n+r-2} + \sum_{n=0}^{\infty} \frac{a_n}{2}\, x^{n+r-2} = 0. \tag{224}$$

Shifting index:

$$\sum_{n=0}^{\infty} (n + r)\, a_n\, x^{n+r-1} = \sum_{n=1}^{\infty} (n + r - 1)\, a_{n-1}\, x^{n+r-2}. \tag{225}$$

Now the equation becomes

$$\left[r\, (r - 1) - \frac{r}{2} + \frac{1}{2} \right] a_0\, x^{r-2} + \sum_{n=1}^{\infty} \left\{ \left[(n + r)\, (n + r - 1) - \frac{1}{2}\, (n + r) + \frac{1}{2} \right] a_n + (n + r - 1)\, a_{n-1} \right\} x^{n+r-2} = 0. \tag{226}$$

The indicial equation is

$$r\,(r-1)-\frac{r}{2}+\frac{1}{2}=0\Longrightarrow r_1=1, r_2=\frac{1}{2}. \tag{227}$$

Their difference is not an integer.

To find y_1 we set $r=r_1=1$. The recurrence relation

$$\left[(n+r)\,(n+r-1)-\frac{1}{2}\,(n+r)+\frac{1}{2}\right]a_n+(n+r-1)\,a_{n-1}=0 \tag{228}$$

becomes

$$\left[n\,(n+1)-\frac{1}{2}\,(n+1)+\frac{1}{2}\right]a_n+n\,a_{n-1}=0 \tag{229}$$

which simplifies to

$$a_n=-\frac{2}{2\,n+1}\,a_{n-1}. \tag{230}$$

This gives

$$a_n=(-1)^n\,\frac{2^n}{(2\,n+1)\,(2\,n-1)\cdots 3}\,a_0. \tag{231}$$

Setting $a_0=1$ we obtain

$$y_1(x)=|x|\sum_{n=0}^{\infty}\,(-1)^n\,\frac{2^n}{(2\,n+1)\,(2\,n-1)\cdots 3}\,x^n. \tag{232}$$

To find y_2 we set $r=r_2=1/2$. The recurrence relation becomes

$$a_n=-\frac{1}{n}\,a_{n-1}\Longrightarrow a_n=(-1)^n\,\frac{1}{n!}\,a_0 \tag{233}$$

so

$$y_2(x)=|x|^{1/2}\sum_{n=0}^{\infty}\,(-1)^n\,\frac{1}{n!}\,x^n=|x|^{1/2}\,e^{-x}. \tag{234}$$

Finall the general solution is

$$y(x)=C_1\,|x|\sum_{n=0}^{\infty}\,(-1)^n\,\frac{2^n}{(2\,n+1)\,(2\,n-1)\cdots 3}\,x^n+C_2\,|x|^{1/2}\,e^{-x}. \tag{235}$$

Remark 48. Of course, for anyone who can remember the formulas, there is no need to do all these differentiation and index-shifting.

Case 2: $r_1=r_2$.

In this case it is clear that $y_1(x)$ can be obtained without any difficulty. The problem is how to obtain $y_2(x)$. The idea is to differentiate $y_1(x)$ with respect to r. To see why this would work, we need to study the dependence of y on r very carefully.

Recall that, when we substitute y by the expansion $x^r\sum a_n\,x^n$ the equation becomes

$$\sum_{n=0}^{\infty}\left\{[(n+r)\,(n+r-1)+(n+r)\,p_0+q_0]\,a_n+\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,a_m\right\}x^{n+r-2}=0. \tag{236}$$

Now we keep the r dependence, set $a_0=1$, and solve a_n through the recurrence relation

$$[(n+r)\,(n+r-1)+(n+r)\,p_0+q_0]\,a_n+\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,a_m=0 \tag{237}$$

for $n=1,2,3,\ldots$. As a consequence, we obtain a_n as functions of r: Denote them by $a_n(r)$. Now define

$$y(x;r)=x^r\left[1+\sum_{n=1}^{\infty}\,a_n(r)\,x^n\right]. \tag{238}$$

We have

$$y''+p(x)\,y'+q(x)\,y=[r\,(r-1)+p_0\,r+q_0]\,x^{r-2}. \tag{239}$$

As we are in the case $r_1 = r_2$, this can be further written as

$$y'' + p(x)\, y' + q(x)\, y = (r - r_1)^2\, x^{r-2}. \tag{240}$$

Now taking partial derivative with respect to r of both sides, we reach

$$\left(\frac{\partial y}{\partial r}\right)'' + p(x)\left(\frac{\partial y}{\partial r}\right)' + q(x)\,\frac{\partial y}{\partial r} = 2\,(r - r_1)\, x^{r-2} + (r - r_1)^2\, x^{r-2} \ln|x|. \tag{241}$$

It is now clear that

$$y_2 = \frac{\partial y}{\partial r}\Big|_{r = r_1} \tag{242}$$

is a solution. As

$$y(x; r) = x^r \left[1 + \sum_{n=1}^{\infty} a_n(r)\, x^n\right], \tag{243}$$

taking r derivative we obtain

$$\frac{\partial y}{\partial r} = \ln|x|\, x^r \left[1 + \sum_{n=1}^{\infty} a_n(r)\, x^n\right] + x^r \sum_{n=1}^{\infty} a_n'(r)\, x^n. \tag{244}$$

consequently

$$y_2(x) = y_1(x) \ln|x| + x^r \sum_{n=1}^{\infty} d_n\, x^n \tag{245}$$

with

$$d_n = a_n'(r_1). \tag{246}$$

Let's work on an example.

Example 49. Solve (at $x_0 = 0$)

$$x\,(1 - x)\, y'' + (1 - x)\, y' - y = 0. \tag{247}$$

Solution. The standard form is

$$y'' + \frac{1}{x}\, y' - \frac{1}{x\,(1 - x)}\, y = 0. \tag{248}$$

It is easy to check that 0 is a regular singular point. We have

$$p(x)\, x = 1 \Longrightarrow p_0 = 1,\ \ p_n = 0 \text{ for all } n \geqslant 1; \tag{249}$$

$$q(x)\, x^2 = -\frac{x}{1 - x} = -x\left(\sum_{n=0}^{\infty} x^n\right) = -\sum_{n=1}^{\infty} x^n \Longrightarrow q_0 = 0,\ \ q_n = -1 \text{ for all } n \geqslant 1. \tag{250}$$

The indicial equation is

$$r\,(r - 1) + p_0\, r + q_0 = 0 \Longrightarrow r\,(r - 1) + r = 0 \Longrightarrow r_1 = r_2 = 0. \tag{251}$$

For y_1 one can easily compute

$$a_n = \frac{(n + r - 1)^2 + 1}{(n + r)^2}\, a_{n-1} \tag{252}$$

which gives

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot 5 \cdots \left((n - 1)^2 + 1\right)}{(n!)^2}\, x^n. \tag{253}$$

To find out y_2 we need to keep the r dependence:

$$a_n = \frac{(r^2 + 1)\left((r + 1)^2 + 1\right) \cdots \left((r + n - 1)^2 + 1\right)}{(r + 1)^2\, (r + 2)^2 \cdots (r + m)^2}\, a_0. \tag{254}$$

The best way to compute a_n' is to take logarithm first:

$$\frac{a_n'}{a_n} = (\ln a_n)' = \sum_{k=1}^{n} \left[\frac{2\,(r+k-1)}{(r+k-1)^2+1} - \frac{2}{r+k} \right]. \tag{255}$$

Setting $r = r_1 = 0$, we get

$$d_n = a_n' = 2\, a_n\,|_{r=0} \sum_{k=1}^{n} \frac{k-2}{k\left((k-1)^2+1\right)} = 2\, \frac{1 \cdot 2 \cdot 5 \cdots \left((n-1)^2+1\right)}{(n!)^2} \sum_{k=1}^{n} \frac{k-2}{k\left((k-1)^2+1\right)}. \tag{256}$$

So the second solution is

$$y_2(x) = y_1(x) \ln|x| + 2 \sum_{n=1}^{\infty} \left[\frac{1 \cdot 2 \cdot 5 \cdots \left((n-1)^2+1\right)}{(n!)^2} \sum_{k=1}^{n} \frac{k-2}{k\left((k-1)^2+1\right)} \right] x^n \tag{257}$$

and the general solution is given by

$$y = C_1\, y_1 + C_2\, y_2. \tag{258}$$

Finally we discuss radius of convergence, that is in which interval is the above solution correct.

Recall that

$$x\,p = 1, \qquad x^2\, q = -\frac{x}{1-x}, \tag{259}$$

we see that there is no singular point for $x\,p$, but $x^2\,q$ becomes singular at $x=1$. The lower bound for the radius of convergence is then $|1-0| = 1$. As a consequence, our solutions are valid at least for $|x| < 1$.

Remark 50. Instead of using the above approach, one can also assume

$$y_2 = y_1 \ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} d_n\, x^n, \tag{260}$$

substitute into the equation, and try to determine d_n. As the existence of a solution of the above form is guaranteed, d_n can always be determined one by one.

Case 3: $r_1 - r_2$ is a non-zero integer.

This is the most complicated case. First let us spend some time understand why $r_1 - r_2$ being an integer is a problem. Recall the indicial equation and the recurrence relation:

$$(n=0): \qquad [r\,(r-1) + p_0\, r + q_0]\, a_0 = 0, \tag{261}$$

$$(n \geqslant 1): \qquad [(n+r)\,(n+r-1) + (n+r)\, p_0 + q_0]\, a_n + \sum_{m=0}^{n-1} [(m+r)\, p_{n-m} + q_{n-m}]\, a_m = 0. \tag{262}$$

We note that, to be able to determine a_n uniquely, we need

$$(n+r)\,(n+r-1) + (n+r)\, p_0 + q_0 \neq 0. \tag{263}$$

This is exactly the same as **$n+r$ is not a solution to the indicial equation**. Therefore, when $r_1 - r_2$ is an integer, if we choose r_1 to be the one with bigger real part, and set $r = r_1$, then we will have no difficulty determining a_n one by one; However, if we set $r = r_2$, the root with smaller real part, the coefficient of a_{n_0} would disappear, where n_0 is such that $r_1 - r_2 = n_0$, thus making determination of a_{n_0} not possible.

Summarizing, when $r_1 - r_2$ is a positive integer, setting $r = r_1$ and calculating a_n one by one would give us the first solution

$$y_1(x) = x^{r_1} \sum a_n\, x^n \tag{264}$$

but setting $r = r_2$ will not give us the linearly independent second solution. We have to find other ways.

Remark 51. (Treating regular points as singular) It is interesting to notice the following. Consider the case 0 is a regular point of

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{265}$$

Now note that 0 also satisfies the definition of regular singular points. Let's pretend we didn't recognize that it is regular, and set

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}. \tag{266}$$

The recurrence relations are then

$$(n=0): \qquad [r(r-1) + p_0 r + q_0] a_0 = 0, \tag{267}$$

$$(n \geqslant 1): \qquad [(n+r)(n+r-1) + (n+r) p_0 + q_0] a_n + \sum_{m=0}^{n-1} [(m+r) p_{n-m} + q_{n-m}] a_m = 0. \tag{268}$$

Now recall that p_n, q_n are from expansions of $x\,p$ and $x^2\,q$. As p, q themselves are analytic, we have $p_0 = q_0 = q_1 = 0$.

The indicial equation now becomes

$$r(r-1) = 0 \Longrightarrow r_1 = 1, r_2 = 0 \tag{269}$$

which is the "scary" case of $r_1 - r_2 = \text{integer}$!

Of course we know that nothing can go wrong and we can determine every a_n one by one. Let's check what saves us.

Taking $r = r_1$ leads to one solution $y_1 = x[\cdots]$. Taking $r = r_2$, the coefficient for a_1 becomes 0 in the recurrence relation for $n = 1$. But taking a closer look reveals that the $n = 1$ relation actually becomes

$$0\,a_1 + [(0 + r_2) p_1 + q_1] a_0 = 0 \tag{270}$$

which is simply

$$0\,a_1 = 0! \tag{271}$$

Therefore, we can simply pick a_1 to be any nonzero number and proceed to determine $a_2, a_3, \ldots$.

This is actually the idea behind the following argument leading to a second solution for the case $r_1 - r_2 = \text{integer}$.

In the following we use r_1, r_2 to denote the two roots, and take r_1 to be the root with larger real part, and denote $n_0 = r_1 - r_2$ to be the positive integer difference between the two roots.

We try to modify the "differentiate with respect to r" trick. Recall that, if we set $a_0 = 1$, and compute a_n as functions of r, then the function $y = y(x; r)$ satisfies

$$y'' + p(x)\,y' + q(x)\,y = [r(r-1) + p_0 r + q_0] x^{r-2} = (r - r_1)(r - r_2) x^{r-2}. \tag{272}$$

Clearly, $\frac{\partial y}{\partial r}$ is not a solution to the original equation no matter what value we assign to r. The reason being that the dependence of the right hand side on $r - r_1$ and $r - r_2$ are both linear.

The fix to this situation comes from the following observation. If instead of setting $a_0 = 1$, we keep the a_0 dependence, $y(x; r)$ would solve

$$y'' + p(x)\,y' + q(x)\,y = a_0 (r - r_1)(r - r_2) x^{r-2}. \tag{273}$$

Now if we take $a_0 = r - r_1$ or $r - r_2$, the right hand side would contain $(r - r_1)^2$ or $(r - r_2)^2$ and the differentiation trick would work.

Say we take $a_0 = r - r_1$. Then

$$y'' + p(x)\,y' + q(x)\,y = (r - r_1)^2 (r - r_2) x^{r-2} \tag{274}$$

and

$$\left(\frac{\partial y}{\partial r}\right)'' + p(x)\left(\frac{\partial y}{\partial r}\right)' + q(x)\left(\frac{\partial y}{\partial r}\right) = \left[2(r - r_1)(r - r_2) + (r - r_1)^2 + (r - r_1)^2 (r - r_2) \ln|x|\right] x^{r-2} \tag{275}$$

Thus $\frac{\partial y}{\partial r}|_{r=r_1}$ solves the equation. Similarly, if we take $a_0 = r - r_2$, $\frac{\partial y}{\partial r}|_{r=r_2}$ would be a solution.

So instead of getting one more solution, we get two more? This sounds too good to be true. Indeed it is. Let's study the two choices more carefully. Let $Y(x;r)$ be the r-dependent solution obtained by setting $a_0=1$. Note that $Y(x;r_2)$ is not well-defined although $Y(x;r)$ for all other r's are. Checking the recurrence relation, we easily see that if we keep a_0 dependence, we would get

$$y(x;r)=a_0\,Y(x;r) \tag{276}$$

except for $r=r_2$.

- Setting $a_0=r-r_1$.

  In this case we get

  $$y(x;r)=(r-r_1)\,Y(x;r) \tag{277}$$

  Compute

  $$\frac{\partial y}{\partial r}|_{r=r_1}=\left[(r-r_1)\,\frac{\partial Y}{\partial r}+Y(x;r)\right]_{r=r_1}=Y(x;r_1). \tag{278}$$

  But $Y(x;r_1)$ is just the solution obtained through setting $r=r_1$ and $a_0=1$! We see that $Y(x;r_1)=y_1(x)$. So we are not getting any new solution!

- Setting $a_0=r-r_2$.

  In this case we cannot simply write $y(x;r)=(r-r_2)\,Y(x;r)$ and differetiate anymore, as $Y(x;r_2)$ is not defined. We need to go deep into the recurrence relation:

  $$[(n+r)\,(n+r-1)+(n+r)\,p_0+q_0]\,a_n+\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,a_m=0. \tag{279}$$

  It can be re-written into

  $$[(n+r-r_1)\,(n+r-r_2)]\,a_n=-\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,a_m. \tag{280}$$

  Divide both sides by the coefficient for a_n, we have

  $$a_n(r)=-\frac{\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,a_m}{(n+r-r_1)\,(n+r-r_2)}. \tag{281}$$

  The denominator would become infinity when $n=n_0$ and $r=r_2$, as the first factor becomes $n_0+r-r_1=r-r_2$. So the difficulty we meet here is: Can we define $y(x;r)$ at $r=r_2$ reasonably?

  Now take into consideration that $a_0=r-r_2$. Clearly, for each $n<n_0$, we can write $a_n(r)=\bar{a}_n(r)\,a_0(r)=\bar{a}_n(r)\,(r-r_2)$, where $\bar{a}_n(r)$ is in fact the value of a_n if we set $a_0=1$. Thus the formula for a_{n_0} becomes

  $$a_{n_0}(r)=-\frac{\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,\bar{a}_m\,(r-r_2)}{(r-r_2)\,(n_0+r-r_2)}=-\frac{\sum_{m=0}^{n-1}\,[(m+r)\,p_{n-m}+q_{n-m}]\,\bar{a}_m}{(n_0+r-r_2)}. \tag{282}$$

  We see that, when we take $a_0=r-r_2$, $a_{n_0}(r)$ is well-defined for all r around r_2 (Keep in mind that all we need is to be able to differentiate new r_2 and then set $r=r_2$.).

  From the above analysis, we see that when taking $a_0=r-r_2$, all coefficients $a_n(r)$ can be determined uniquely.

  Now let

  $$y(x;r)=x^r\sum\,a_n(r)\,x^n. \tag{283}$$

  We see that

  $$y_2(x)=\frac{\partial y(x;r)}{\partial r}|_{r=r_2}=x^{r_2}\ln|x|\sum_{n=0}^{\infty}\,a_n(r_2)\,x^n+x^{r_2}\sum_{n=0}^{\infty}\,a_n'(r_2)\,x^n. \tag{284}$$

Finally, we notice that, $a_n(r_2)=0$ for all $n<n_0$, but $a_{n_0}(r_2)\neq 0$. As a consequence, $a_n(r_2)\neq 0$ when $n>n_0$ as well.

So the first term is in fact

$$x^{r_2}\ln|x|\sum_{n=n_0}^{\infty} a_n(r_2)\,x^n = x^{r_1}\ln|x|\sum_{n=0}^{\infty} a_{n+n_0}(r_2)\,x^n. \tag{285}$$

To further simplify, we again study the recurrence relation.

$$(n=0): \qquad [r\,(r-1)+p_0\,r+q_0]\,a_0 = 0, \tag{286}$$

$$(n\geqslant 1): \qquad [(n+r)\,(n+r-1)+(n+r)\,p_0+q_0]\,a_n+\sum_{m=0}^{n-1}[(m+r)\,p_{n-m}+q_{n-m}]\,a_m = 0. \tag{287}$$

We know that $a_n(r_2)=0$ for every $n<n_0$. So the recurrence relation for $n>n_0$ can be written as

$$[(n+r)\,(n+r-1)+(n+r)\,p_0+q_0]\,a_n+\sum_{m=n_0}^{n-1}[(m+r)\,p_{n-m}+q_{n-m}]\,a_m=0. \tag{288}$$

Now define $\tilde{r}(r)=r+n_0$, let $\tilde{n}=n-n_0$, and $\tilde{a}_{\tilde{n}}=a_n$. We see that $\tilde{r}(r_2)=r_1$. The recurrence relation now becomes

$$(\tilde{n}\geqslant 1): \qquad [(\tilde{n}+\tilde{r})\,(\tilde{n}+\tilde{r}-1)+(\tilde{n}+\tilde{r})\,p_0+q_0]\,\tilde{a}_{\tilde{n}}+\sum_{m=0}^{\tilde{n}-1}[(m+\tilde{r})\,p_{\tilde{n}-m}+q_{\tilde{n}-m}]\,\tilde{a}_m=0. \tag{289}$$

This is exactly the same as the original recurrence relation! As a consequence we have

$$\tilde{a}_n(r_2)=a_n(r_1). \tag{290}$$

As a consequence,

$$x^{r_1}\ln|x|\sum_{n=0}^{\infty} a_{n+n_0}(r_2)\,x^n = x^{r_1}\ln|x|\sum_{n=0}^{\infty}\tilde{a}_n(r_2)\,x^n = a_{n_0}(r_2)\,x^{r_1}\ln|x|\sum_{n=0}^{\infty}a_n(r_1)\,x^n = a_{n_0}(r_2)\ln|x|\,y_1. \tag{291}$$

We see that the constant C in the theorem is exactly $a_{n_0}(r_2)$.

Example 52. Solve

$$x\,y''+2\,y'-y=0 \tag{292}$$

at 0.

Solution. First write it as

$$y''+\frac{2}{x}\,y'-\frac{1}{x}\,y=0. \tag{293}$$

It is clear that $x=0$ is a regular singular point, with $x\,p=2$ and $x^2\,q=-x$. That is, $p_0=2$, $q_1=-1$, and all other p_n, q_n are 0.

The indicial equation is

$$r\,(r-1)+2\;r=0\Longrightarrow r_1=0, r_2-1. \tag{294}$$

Note that we have already taken r_1 to be the one with larger real part.

The recurrence relation for $n\geqslant 1$ is

$$(n+r)\,(n+r+1)\,a_n=a_{n-1}, \qquad n=1,2,\ldots. \tag{295}$$

Taking $r=r_1=0$, we reach

$$a_n=\frac{1}{n!\,(n+1)!}\,a_0 \tag{296}$$

which gives the first solution

$$y_1(x)=\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}\,x^n. \tag{297}$$

Now we find the second solution. Take $a_0=r-r_2=r+1$. Then setting $n=1$ in the recurrence relation we obtain

$$(r+1)\,(r+2)\,a_1=a_0=r+1\Longrightarrow a_1=\frac{1}{r+2}. \tag{298}$$

As $a_1(r_2) = \frac{1}{-1+2} = 1$, the constant $C = 1$.

The recurrence relation then gives

$$a_n = \frac{1}{(r+2)^2 \cdots (r+n)^2 \, (r+n+1)}. \tag{299}$$

Taking logarithm and differentiate, we reach

$$\frac{a_n'}{a_n} = -2 \sum_{k=2}^{n} \frac{1}{r+k} - \frac{1}{r+n+1}, \qquad n = 1, 2, 3, \ldots. \tag{300}$$

Now we can write down the second solution

$$y_2(x) = y_1(x) \ln|x| + |x|^{-1} \left[1 - \sum_{n=1}^{\infty} \frac{1}{n! \, (n-1)!} \left(2 \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n} \right) x^n \right]. \tag{301}$$

The general solution, of course, is

$$y = C_1 \, y_1 + C_2 \, y_2. \tag{302}$$

This formula is valid for $0 < |x| < \infty$.

4. Special Functions.

From the differential equation point of view, “special functions” are solutions to particular classes of equations, each involving a parameter. The most popular “special functions” are the trignometric functions $\cos(n\, x)$ and $\sin(n\, x)$, which are solutions to the boundary value problems

$$y'' + n^2 \, y = 0, \qquad y(0) = y(2\,\pi), \quad y'(0) = y'(2\,\pi). \tag{303}$$

The importance of cos and sin in both theory and practice comes from the following fact:

For any function f defined on $0 \leqslant x \leqslant 2\,\pi$, there are constants a_n and b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\, x + b_n \sin n\, x]. \tag{304}$$

Here the equality “$=$” may differ slightly from “equals everywhere”.

It turns out that, for many other second order differential equations with one or more parameters, the solutions to their appropriate boundary value problems (called “special functions”) have similar ability of representing general functions. This is the motivation behind studying such “special functions”.

Remark 53. There are many other ways to introduce special functions. One particularly interesting approach is through representation of Lie groups.

4.1. Bessel functions.

Bessel functions are solutions to the equation

$$x^2 \, y'' + x \, y' + \left(x^2 - \nu^2 \right) y = 0 \tag{305}$$

where ν is a parameter. Writing the equation into standard form, we have

$$y'' + \frac{1}{x} \, y' + \frac{x^2 - \nu^2}{x^2} \, y = 0. \tag{306}$$

It is clear that $x = 0$ is a regular singular point. And furthermore we have

$$p_0 = 1, \quad q_0 = -\nu^2, \quad q_2 = 1, \qquad p_n = q_n = 0 \text{ for all other } n. \tag{307}$$

The indicial equation is

$$r\,(r-1) + r - \nu^2 = 0 \Longrightarrow r_{1,2} = \pm\,\nu. \tag{308}$$

If we consider the case ν is a non-negative integer, then $r_1 = \nu, r_2 = -\nu$.

Now we need to discuss case by case.

- $r_1 - r_2 = 2\,\nu$ is not an integer.
  Using the recurrence relation we obtain
$$y_1(x) = \left[1 - \frac{1}{2^2\,(1+\nu)\,1!}\,x^2 + \frac{1}{2^4\,(1+a)\,(2+a)\,2!}\,x^4 - \cdots\right] x^{\nu} \tag{309}$$
  and
$$y_2(x) = \left[1 - \frac{1}{2^2\,(1-a)\,1!}\,x^2 + \frac{1}{2^4\,(1-a)\,(2-a)\,2!}\,x^4 - \cdots\right] x^{-\nu}. \tag{310}$$

- $\nu = 0$, that is $r_1 = r_2$.
  In this case we can take
$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \tag{311}$$
  and for the second solution we compute
$$a_{2n} = \frac{(-1)^n}{(r+2)^2\,(r+4)^2 \cdots (r+2\,n)^2} \tag{312}$$
  which gives
$$y_2(x) = y_1(x)\ln|x| - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{k=1}^{n} \frac{1}{k}\right) \left(\frac{x}{2}\right)^{2n}. \tag{313}$$

- $2\,\nu$ is an integer. This is further divided into two cases.
  - ν is not an integer.
    In this we have
$$y(x) = x^{-\nu}\left[C_1 \sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(1-\nu)}{n!\,2^{2m}\,\Gamma(1-\nu+n)}\,x^{2n} + C_2 \sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(\nu+1)}{m!\,2^{2m}\,\Gamma(n+\nu+1)}\,x^{2\nu+2m}\right]. \tag{314}$$
  - ν is an integer.
    In this case
$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu}, \tag{315}$$
    while y_2 takes the form
$$\frac{2}{\pi}\left[\left(\gamma + \ln\left|\frac{x}{2}\right|\right) y_1 - \frac{1}{2}\sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-\nu} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^{n+1} \frac{\phi(n)+\phi(n+\nu)}{n!\,(n+\nu)!} \left(\frac{x}{2}\right)^{2n+\nu}\right]. \tag{316}$$
    Here $\phi(0) = 0$, $\phi(n) = \sum_{k=1}^{n} (1/k)$ and γ is the Euler constant:
$$\gamma = \lim_{n \nearrow \infty} (\phi(n) - \ln n) = 0.5772157\cdots. \tag{317}$$

5. Beyond regular singular point.

What happens when the singular point is not regular? One major change is that it is called a **irregular singular point** instead.

Anyway, we briefly mention a few things.

- The ansatz
$$y = x^r \sum a_n\,x^n \tag{318}$$
  may not yield any solution.

Example 54. Solving $x^3\, y'' - y = 0$ using Frobenius' method would lead to $a_0 = 0$, which is not allowed.

Example 55. Solving $x^2\, y'' + (1 + 3\,x)\, y' + y = 0$ using Frobenius' method leads to

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n\, n!\, x^n \tag{319}$$

which is a useless formula as the radius of convergence is 0.

- There is no complete theory. One has to be very clever at guessing the form of the solutions, and be very good in analysis to show convergence of the formal sum obtained. Of course, after a few geniuses have worked on such equations, some "rules of thumb" become available. Nevertheless, solving these equations remains largely an art.
- Dealing with irregular singular points is one major topic in a branch of mathematics called "asymptotic analysis".