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The Method of Laplace Transform

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1.1. Definition of Laplace transform.

Definition 1. *Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral*

$$F(s) := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t. \tag{1}$$

Remark 2. Clearly $F(s)$ may not exist for all s – that is, the integral may be either infinite or not well-defined. Those values of s for which the integral exists is called the domain of $F(s)$.

Remark 3. Often $\mathcal{L}\{f\}$ is used instead of $F(s)$.

Remark 4. Why would we introduce Laplace transform? The reason is that, through Laplace transform, a differential equation becomes an algebraic equation. We will discuss this systematically later, but let's look at an example to get some idea now.

Example 5. Compute that Laplace transform of $y' = f(t)$.
Solution. The transformed equation should be

$$\mathcal{L}\{y'\}(s) = \mathcal{L}\{f\}(s). \tag{2}$$

We now try to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$. By definition

$$\begin{aligned} \mathcal{L}\{y'\}(s) &= \int_0^\infty e^{-st}\, y'(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{-st}\,\mathrm{d}y(t) \\ &= e^{-st}\, y(t)\,|_0^\infty - \int_0^\infty y(t)\,\mathrm{d}e^{-st} \\ &= -y(0) + s\int_0^\infty e^{-st}\, y(t)\,\mathrm{d}t \\ &= -y(0) + s\,\mathcal{L}\{y\}(s). \end{aligned} \tag{3}$$

Note that in the above we have assumed

$$\lim_{t\nearrow\infty}\left[e^{-st}\,y(t)\right]=0. \tag{4}$$

Thus the equation $y'=f$ is transformed into

$$s\,\mathcal{L}\{y\}(s)-y(0)=\mathcal{L}\{f\}(s)\Longrightarrow\mathcal{L}\{y\}(s)=\frac{\mathcal{L}\{y\}(s)+y(0)}{s}. \tag{5}$$

We see that no integration is performed and we have found $\mathcal{L}\{y\}$! Thus as long as we can "invert" the transform, we will get the solution y without any integration.

Of course, we see that the success of this program depends on two things:

- Our ability to transform: Obtain $\mathcal{L}\{f\}$ from f, and
- out ability to invert: Obtain y from $\mathcal{L}\{y\}$.

Computing Laplace transforms is a good practice of integration techniques.

Example 6. Compute the Laplace transform of the following functions.

$$f(t)=1, e^{at}, t^n, \sin b\,t, \cos b\,t, e^{at}\,t^n, e^{at}\sin b\,t, e^{at}\cos b\,t. \tag{6}$$

Solution.

1. $f(t)=1$. We compute

$$\mathcal{L}\{f\}(s)=\int_0^\infty e^{-st}\,\mathrm{d}t. \tag{7}$$

   Clearly the integral is not finite for $s\leqslant 0$. For $s\geqslant 0$, We have

$$\mathcal{L}\{f\}(s)=\int_0^\infty e^{-st}\,\mathrm{d}t=-\frac{1}{s}\,e^{-st}\,|_0^\infty=\frac{1}{s}. \tag{8}$$

2. $f(t)=e^{at}$. We compute

$$\mathcal{L}\{f\}(s)=\int_0^\infty e^{at}\,e^{-st}\,\mathrm{d}t=\int_0^\infty e^{(a-s)t}\,\mathrm{d}t=\frac{1}{s-a}. \tag{9}$$

   The domain is $s>a$.

3. $f(t)=t^n$, $n=1,2,\ldots.$ Clearly we need to require $s>0$, otherwise the integral is not finite.

   Compute

$$\begin{aligned}\mathcal{L}\{t^n\}(s) &= \int_0^\infty t^n\,e^{-st}\,\mathrm{d}t\\ &= -\frac{1}{s}\int_0^\infty t^n\,\mathrm{d}e^{-st}\\ &= -\frac{1}{s}\,t^n\,e^{-st}\,|_0^\infty+\frac{1}{s}\int e^{-st}\,\mathrm{d}t^n\\ &= \frac{n}{s}\int t^{n-1}\,e^{-st}\,\mathrm{d}t\\ &= \frac{n}{s}\,\mathcal{L}\{t^{n-1}\}(s).\end{aligned} \tag{10}$$

   Replacing n by $n-1$ we have

$$\mathcal{L}\{t^{n-1}\}(s)=\frac{n-1}{s}\,\mathcal{L}\{t^{n-2}\}(s). \tag{11}$$

   Thus we have

$$\mathcal{L}\{t^n\}(s)=\frac{n}{s}\,\mathcal{L}\{t^{n-1}\}(s)=\frac{n\,(n-1)}{s^2}\,\mathcal{L}\{t^{n-2}\}(s)=\cdots=\frac{n!}{s^n}\,\mathcal{L}\{t^0\}(s)=\frac{n!}{s^{n+1}}. \tag{12}$$

   The domain is $s>0$.

4. $f(t) = \sin b\,t$. Again we need to require $s > 0$ as otherwise the integral does not exist. We compute

$$\begin{aligned}
\mathcal{L}\{\sin b\,t\}(s) &= \int_0^\infty \sin b\,t\, e^{-st}\,\mathrm{d}t \\
&= -\frac{1}{s}\int_0^\infty \sin b\,t\,\mathrm{d}e^{-st} \\
&= -\frac{1}{s}\sin b\,t\,e^{-st}|_0^\infty + \frac{1}{s}\int e^{-st}\,\mathrm{d}\sin b\,t \\
&= 0 + \frac{b}{s}\int e^{-st}\cos b\,t\,\mathrm{d}t \\
&= -\frac{b}{s^2}\int \cos b\,t\,\mathrm{d}e^{-st} \\
&= -\frac{b}{s^2}\left[\cos b\,t\,e^{-st}|_0^\infty - \int e^{-st}\,\mathrm{d}\cos b\,t\right] \\
&= -\frac{b}{s^2}\left[-1 + b\int e^{-st}\sin b\,t\right] \\
&= \frac{b}{s^2} - \frac{b^2}{s^2}\mathcal{L}\{\sin b\,t\}(s). \qquad (13)
\end{aligned}$$

This gives

$$\mathcal{L}\{\sin b\,t\}(s) = \frac{b}{s^2 + b^2}, \qquad s > 0. \qquad (14)$$

5. $f(t) = \cos\, b\,t$. We can proceed similarly. But a quicker way is to notice that in the calculation of $\mathcal{L}\{\sin b\,t\}(s)$ we already obtain

$$\mathcal{L}\{\sin b\,t\}(s) = \frac{b}{s}\int e^{-st}\cos b\,t\,\mathrm{d}t = \frac{b}{s}\mathcal{L}\{\cos b\,t\}(s). \qquad (15)$$

Thus

$$\mathcal{L}\{\cos b\,t\}(s) = \frac{s}{s^2 + b^2}, \qquad s > 0. \qquad (16)$$

6. $f(t) = e^{at}\,t^n$, $n = 1, 2, \ldots$ We can compute using definition, but a quicker way is to notice that

$$\mathcal{L}\left\{e^{at}\,t^n\right\}(s) = \int_0^\infty e^{-(s-a)t}\,t^n\,\mathrm{d}t. \qquad (17)$$

This is exactly the formula for $\mathcal{L}\{t^n\}$ with s replaced by $s - a$. Replacing every s by $s - a$ in the t^n case, we have

$$\mathcal{L}\left\{e^{at}\,t^n\right\}(s) = \mathcal{L}\{t^n\}(s - a) = \frac{n!}{(s - a)^{n+1}}. \qquad (18)$$

Naturally, the domain changes from $s > 0$ to $s - a > 0$, or $s > a$.

7. $f(t) = e^{at}\sin b\,t$. Similarly, we conclude

$$\mathcal{L}\left\{e^{at}\sin b\,t\right\}(s) = \mathcal{L}\{\sin b\,t\}(s - a) = \frac{b}{(s - a)^2 + b^2} \qquad (19)$$

with domain $s > a$.

8. $f(t) = e^{at}\cos b\,t$. Similarly we obtain

$$\mathcal{L}\left\{e^{at}\cos b\,t\right\} = \frac{s}{(s - a)^2 + b^2}. \qquad (20)$$

1.2. Some theoretical issues.

As we have mentioned, for a function f, its Laplace transform may not exist for all s. The following theorem tells us for what s the transform exists.

Theorem 7. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.*

Remark 8. A function f is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has jump discontinuity[1]. A function f is said to be piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Remark 9. A function f is said to be of exponential order α if there exist positive constant T and M such that

$$|f(t)| \leqslant M e^{\alpha t}, \qquad \text{for all } t \geqslant T. \tag{21}$$

A function f is of "exponential order" if there is α such that f is of exponential order α.

Example 10. Check whether

$$f(t) = \begin{cases} 1 & 0 \leqslant t < 1 \\ t - 1 & 1 < t < 3 \\ t^2 - 4 & 3 < t \leqslant 10 \end{cases} \tag{22}$$

is piecewise continuous on $[0, 10]$.

Solution. Clearly f is continuous at all points other than $t = 1, t = 3$. All we need to do is to check that at $t = 1, 3$,

- is f continuous? If yes, then f is continuous on $[0, 10]$ and thus piecewise continuous.
- If not, are the discontinuities "jump discontinuities"? If yes, then f is piecewise continuous.
- If no, then f is not piecewise continuous.

We easily check that f is not continuous at $1, 3$ but the discontinuities are "jump discontinuities". Thus f is piecewise continuous on $[0, 10]$.

Example 11. Is $t^3 \sin t$ of exponential order?

Solution. We need to check whether there is α, M and T such that

$$\left|t^3 \sin t\right| \leqslant M e^{\alpha t}, \qquad \text{for all } t \geqslant T. \tag{23}$$

As $\left|t^3 \sin t\right| \leqslant \left|t^3\right|$ and

$$\frac{\left|t^3\right|}{e^{\alpha t}} \longrightarrow 0 \qquad \text{as } t \nearrow \infty \tag{24}$$

for any $\alpha > 0$, we see that for any $\alpha > 0$, there are M, T such that

$$\left|t^3 \sin t\right| \leqslant M e^{\alpha t}, \qquad \text{for all } t \geqslant T. \tag{25}$$

As a consequence, $t^3 \sin t$ is of exponential order.

1.3. Properties of Laplace transform.

Recall that, to use Laplace transform in solving ODEs with constant-coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \tag{26}$$

We need to be able to

1. Transform the DE into an algebraic equation. Or more specifically, transform derivatives.
2. Compute $\mathcal{L}\{f\}$ for general f – or at least a wide class of functions.
3. Obtain y from $\mathcal{L}\{y\}$.

In the following (and the next section) we study these three issues in detail.

Laplace transform of derivatives.

We have already shown that

$$\mathcal{L}\{y'\}(s) = s \mathcal{L}\{y\}(s) - y(0). \tag{27}$$

1. t_0 is a "jump discontinuity" if f is not continuous at t_0, but the two one-sided limits both exist. In other words, t_0 is a "jump discontinuity" if the two one-sided limits exist but are not equal.

In general, we have

Theorem 12. *Let $y(t), y'(t), \ldots, y^{(n-1)}(t)$ be continuous on $[0,\infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0,\infty)$, with all these functions of exponential order α. Then for $s>\alpha$,*

$$\mathcal{L}\left\{y^{(n)}\right\}(s)=s^n\,\mathcal{L}(y)(s)-s^{n-1}\,y(0)-s^{n-2}\,y'(0)-\cdots-y^{(n-1)}(0). \tag{28}$$

Example 13. The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, when all initial conditions are zero. If a linear system is governed by the differential equation

$$y''+6\,y'+10\,y=g, \qquad t>0, \tag{29}$$

determine its transfer function.

Solution. We compute

$$\mathcal{L}\{y''\}=s^2\,\mathcal{L}\{y\}(s)-s\,y(0)-y'(0)=s^2\,\mathcal{L}\{y\}; \tag{30}$$

$$\mathcal{L}\{y'\}=s\,\mathcal{L}\{y\}-y(0)=s\,\mathcal{L}\{y\}. \tag{31}$$

Thus

$$\mathcal{L}\{y''+6\,y'+10\,y\}=\mathcal{L}\{y''\}+6\,\mathcal{L}\{y'\}+10\,\mathcal{L}\{y\}=\left(s^2+6\,s+10\right)\mathcal{L}\{y\}. \tag{32}$$

This leads to

$$\left(s^2+6\,s+10\right)\mathcal{L}\{y\}=\mathcal{L}\{g\}\Longrightarrow\frac{\mathcal{L}\{y\}}{\mathcal{L}\{g\}}=\frac{1}{s^2+6\,s+10}. \tag{33}$$

which is the transfer function for this linear system.

Computing Laplace transforms.

In the above we have computed Laplace transform of several popular functions: t^n, $e^{a t}$, $\sin b\,t$, $\cos b\,t$. The following properties allow us to obtain Laplace transform of functions constructed using these “basic“ functions.

1. Linearity.

   Theorem 14. *Let f_1, f_2 be functions whose Laplace transforms exist for $s>\alpha$ and let c_1, c_2 be constants. Then for $s>\alpha$,*

   $$\mathcal{L}\{c_1\,f_1+c_2\,f_2\}=c_1\,\mathcal{L}\{f_1\}+c_2\,\mathcal{L}\{f_2\}. \tag{34}$$

   Example 15. Compute

   $$\mathcal{L}\left\{5-e^{2t}+6\,t^2\right\}. \tag{35}$$

   Solution. Clearly we should use linearity and write

   $$\mathcal{L}\left\{5-e^{2t}+6\,t^2\right\}=\mathcal{L}\left\{5\cdot1+(-1)\,e^{2t}+6\,t^2\right\}=5\,\mathcal{L}\{1\}+(-1)\,\mathcal{L}\left\{e^{2t}\right\}+6\,\mathcal{L}\left\{t^2\right\}. \tag{36}$$

   Recall

   $$\mathcal{L}\{1\}=\frac{1}{s},\quad \mathcal{L}\left\{e^{a t}\right\}=\frac{1}{s-a},\quad \mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}. \tag{37}$$

   We have

   $$\mathcal{L}\left\{5-e^{2t}+6\,t^2\right\}=\frac{5}{s}-\frac{1}{s-2}+\frac{12}{s^3}. \tag{38}$$

2. Laplace transform of products.

   In general, computing the Laplace transform of $f\,g$ knowing $\mathcal{L}(f)$, $\mathcal{L}(g)$ is quite difficult. However, when one of f, g is of the form $e^{a t}$ or t^n, there are simple rules.

   a. $e^{a t}$. We have

   $$\mathcal{L}\left\{e^{a t}\,f(t)\right\}(s)=\mathcal{L}\{f\}(s-a). \tag{39}$$

If the domain for $\mathcal{L}\{f\}$ is $s>\alpha$, then the domain for $\mathcal{L}\{e^{at} f\}$ is $s>a+\alpha$.

Proof. The proof is quite easy. By definition

$$\begin{aligned} \mathcal{L}\{e^{at} f(t)\}(s) &= \int_0^\infty e^{-st} e^{at} f(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t \\ &= \mathcal{L}\{f\}(s-a). \end{aligned} \tag{40}$$

□

We look at an example.

Example 16. Compute the Laplace transform of $e^{7t}\sin^2 t$.
Solution. We see that we only need to compute $\mathcal{L}\{\sin^2 t\}$ as

$$\mathcal{L}\{e^{7t}\sin^2 t\}(s) = \mathcal{L}\{\sin^2 t\}(s-7). \tag{41}$$

To compute $\mathcal{L}\{\sin^2 t\}$, we use the trignometric relation

$$1-2\sin^2 t = \cos 2t \Longrightarrow \sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos 2t. \tag{42}$$

Thus

$$\begin{aligned} \mathcal{L}\{\sin^2 t\} &= \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos 2t\right\} \\ &= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\} \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4}. \end{aligned} \tag{43}$$

So finally we have

$$\mathcal{L}\{e^{7t}\sin^2 t\}(s) = \mathcal{L}\{\sin^2 t\}(s-7) = \frac{1}{2(s-7)} - \frac{1}{2}\frac{s-7}{(s-7)^2+4}. \tag{44}$$

b. t^n. We have

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}(\mathcal{L}\{f\}(s)). \tag{45}$$

Proof. For this one, it is easier to derive the LHS from the RHS. By definition we have

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t. \tag{46}$$

Taking derivative:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{f\}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]\,\mathrm{d}t = \int_0^\infty e^{-st}(-t) f(t)\,\mathrm{d}t. \tag{47}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{f\}(s) = \mathcal{L}\{(-t) f\}(s). \tag{48}$$

Taking one more derivative, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}\{f\}(s) = \frac{\mathrm{d}}{\mathrm{d}s}[\mathcal{L}\{(-t) f\}(s)] = \mathcal{L}\left\{(-t)^2 f\right\}(s). \tag{49}$$

In general, if we have

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}(\mathcal{L}\{f\}(s)) = \mathcal{L}\left\{(-t)^k f\right\}(s), \tag{50}$$

Then

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}}(\mathcal{L}\{f\}(s)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\Big\{(-t)^k f\Big\}(s)\Big) = \mathcal{L}\Big\{(-t)^{k+1} f\Big\}(s). \tag{51}$$

Thus we have

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}(\mathcal{L}\{f\}(s)) = \mathcal{L}\{(-t)^n f\}(s) \tag{52}$$

which is equivalent to what we need to prove. □

Example 17. Starting with the transform $\mathcal{L}\{1\}(s) = 1/s$, show that $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$.
Solution. Clearly

$$\mathcal{L}\{t^n\}(s) = \mathcal{L}\{t^n \cdot 1\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{1\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}}. \tag{53}$$

Example 18. Compute

$$\mathcal{L}\big\{e^{-t}\, t \sin 2\,t\big\}(s). \tag{54}$$

Solution. We have

$$\begin{aligned} \mathcal{L}\big\{e^{-t}\, t\sin 2\,t\big\}(s) &= \mathcal{L}\{t \sin 2\,t\}(s+1) \\ &= (-1)\left[\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin 2\,t\}\right](s+1) \\ &= -\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right)\right](s+1) \\ &= F(s+1) \end{aligned} \tag{55}$$

where

$$F(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right) = \frac{4\,s}{(s^2+4)^2}. \tag{56}$$

So

$$\mathcal{L}\big\{e^{-t}\, t\sin 2\,t\big\}(s) = \frac{4\,(s+1)}{\big[(s+1)^2+4\big]^2}. \tag{57}$$

1.4. Inverse Laplace transform.

Recall that to solve a linear constant-coefficient ODE via Laplace transform, we first transform both sides of the equation, then solve the resulting linear equation to obtain $\mathcal{L}\{y\}$, and finally transform "back" to obtain y. Thus we need to

1. be able to transform the equation;

2. be able to transform "back" to obtain a function knowing its Laplace transform.

In the previous section we have dealt with the first issue. Now we turn to the second.

Definition 19. *Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies*

$$\mathcal{L}\{f\} = F \tag{58}$$

*then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.*

The basic method of inversing Laplace transforms is to use the table of Laplace transforms as well as the properties of Laplace transform.

More specifically,

$$\mathcal{L}\{1\} = \frac{1}{s} \implies \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{59}$$

$$\mathcal{L}\{e^{at} f\} = \mathcal{L}\{f\}(s-a) \implies \mathcal{L}^{-1}\{\mathcal{L}\{f\}(s-a)\} = e^{at} f, \tag{60}$$

$$\mathcal{L}\{\sin b\,t\} = \frac{b}{s^2+b^2} \implies \mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin b\,t, \tag{61}$$

$$\mathcal{L}\{\cos b\,t\} = \frac{s}{s^2+b^2} \implies \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos b\,t, \tag{62}$$

$$\mathcal{L}\{t^n f\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f\} \implies \mathcal{L}^{-1}\left\{(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f\}\right\} = t^n f \tag{63}$$

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1 \mathcal{L}\{f\} + c_2 \mathcal{L}\{g\} \implies \mathcal{L}^{-1}\{c_1 F + c_2 G\} = c_1 \mathcal{L}^{-1}\{F\} + c_2 \mathcal{L}^{-1}\{G\}. \tag{64}$$

Example 20. Find

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2\,s+10}\right\}. \tag{65}$$

Solution. We notice

$$\frac{s+1}{s^2+2\,s+10} = \frac{s+1}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+3^2}. \tag{66}$$

Thus recalling the transform formulas for $e^{at} f$ and for $\cos b\,t$, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2\,s+10}\right\} &= e^{-t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} \\ &= e^{-t}\cos 3\,t.\end{aligned} \tag{67}$$

Example 21. Find

$$\mathcal{L}^{-1}\left\{\frac{3}{(2\,s+5)^3}\right\}. \tag{68}$$

Solution. We have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{(2\,s+5)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{2^3\left(s+\frac{5}{2}\right)^3}\right\} \\ &= \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^3}\right\} \\ &= \frac{3}{8} e^{-\frac{5}{2}t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \\ &= \frac{3}{16} e^{-\frac{5}{2}t}\,t^2.\end{aligned} \tag{69}$$

Example 22. Find

$$\mathcal{L}^{-1}\left\{\frac{s-1}{2\,s^2+s+6}\right\}. \tag{70}$$

Solution. Note that the denominator is not of the form $(s-1)^2+b^2$. But we can write

$$\begin{aligned}\frac{s-1}{2\,s^2+s+6} &= \frac{1}{2}\left[\frac{s-1}{s^2+s/2+3}\right] \\ &= \frac{1}{2}\left[\frac{s-1}{(s+1/4)^2+47/16}\right] \\ &= \frac{1}{2}\frac{s+1/4}{(s+1/4)^2+47/16} - \frac{1}{2}\frac{5/4}{(s+1/4)^2+47/16} \\ &= \frac{1}{2}\frac{s+1/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2} - \frac{5}{2\sqrt{47}}\frac{\sqrt{47}/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}.\end{aligned} \tag{71}$$

Thus

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+s+6}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+1/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}\right\}-\frac{5}{2\sqrt{47}}\mathcal{L}^{-1}\left\{\frac{\sqrt{47}/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}\right\}\\
&= \frac{1}{2}e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{47}}{4}t\right)-\frac{5}{2\sqrt{47}}e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{47}}{4}t\right).
\end{aligned}
\tag{72}
$$

In practice, one often needs to invert rational functions, that is functions of the type $F(s)=\frac{P(s)}{Q(s)}$ where P, Q are polynomials, with the degree of P less than that of Q. [2] There is a systematical method, called "method of partial fractions", inverting such rational functions.

The basic idea is to write $\frac{P}{Q}$ into the sum of functions of the type $\frac{A}{(s-r)^m}$. As

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{A}{(s-r)^m}\right\} &= A\,e^{rt}\,\mathcal{L}^{-1}\left\{\frac{1}{s^m}\right\}\\
&= A\,e^{rt}\,\mathcal{L}^{-1}\left\{(-1)^{m-1}\frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(\frac{1}{s}\right)\right\}\\
&= A\,e^{rt}\,\frac{1}{(m-1)!}t^{m-1}
\end{aligned}
\tag{73}
$$

we can then invert each term and obtain $\mathcal{L}^{-1}\{P/Q\}$.

To carry out this plan, we need to first factorize Q:

$$Q(s)=(s-r_1)\cdots(s-r_n) \tag{74}$$

where n is the degree of Q. Clearly there are three cases:

1. All r_i's are real and distinct;
2. Some r_i's are real and repeated;
3. Some r_i's are complex.

We discuss them one by one.

1. All r_i's are real and distinct.

   In this case we can find appropriate constants $A_1,\ldots,A_n$ such that

   $$\frac{P(s)}{Q(s)}=\frac{A_1}{s-r_1}+\cdots+\frac{A_n}{s-r_n}. \tag{75}$$

   Example 23. Compute

   $$\mathcal{L}^{-1}\left\{\frac{6s^2-13s+2}{s(s-1)(s-6)}\right\}. \tag{76}$$

   Solution. First we check that the degree of the denominator is indeed higher than the degree of the nominator. Thus we can write

   $$\frac{6s^2-13s+2}{s(s-1)(s-6)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-6}. \tag{77}$$

   Summing the RHS gives

   $$\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-6} = \frac{A(s-1)(s-6)+B\,s(s-6)+C\,s(s-1)}{s(s-1)(s-6)} \tag{78}$$

   We need to find A,B,C such that

   $$A(s-1)(s-6)+B\,s(s-6)+C\,s(s-1)=6s^2-13s+2. \tag{79}$$

2. If that is not the case, we can always write $P(s)=P'(s)\,Q(s)+R(s)$ where P',R are also polynomials, with the degree of R less than that of Q.

Naïvely, one may want to expand the LHS into

$$(A+B+C)\,s^2+(-7\,A-6\,B-C)\,s+6\,A \tag{80}$$

and then solve

$$\begin{aligned} A+B+C &= 6 && (81)\\ -7\,A-6\,B-C &= -13 && (82)\\ 6\,A &= 2. && (83) \end{aligned}$$

However there is a much simpler way. The key observation is that when we set $s=0,1,6$, exactly two of the three terms vanish. In other words, when we set $s=0,1,6$, exactly one unknown is left in the equation – one equation, one unknown, linear: the simplest equation possible!

- Setting $s=0$, we have
$$A\,(0-1)\,(0-6)=2\Longrightarrow A=1/3. \tag{84}$$
- Setting $s=1$, we have
$$B\,(1-6)=-5\Longrightarrow B=1. \tag{85}$$
- Setting $s=6$, we have
$$C\,6\,(6-1)=216-78+2=140\Longrightarrow C=14/3. \tag{86}$$

Thus the solution is

$$A=\frac{1}{3},\quad B=1,\quad C=\frac{14}{3}. \tag{87}$$

Therefore we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{6\,s^2-13\,s+2}{s\,(s-1)\,(s-6)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1/3}{s}+\frac{1}{s-1}+\frac{14/3}{s-6}\right\}\\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{14}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}\\ &= \frac{1}{3}+e^t+\frac{14}{3}\,e^{6t}. \end{aligned} \tag{88}$$

2. Some r_i's are real and repeated.

   In this case, for this particular r_i, we need to put int

$$\frac{A_{i1}}{s-r_i}+\cdots+\frac{A_{im}}{(s-r_i)^m} \tag{89}$$

where m is the multiplicity of this particular r_i.

Example 24. Compute

$$\mathcal{L}^{-1}\left\{\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}\right\}. \tag{90}$$

Solution. Again, we first check that the nominator's degree is lower.

Next we write the function into partial fractions:

$$\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}=\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{s+1}. \tag{91}$$

Calculating the RHS, we have

$$\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{s+1} = \frac{A\,(s+3)\,(s+1)+B\,(s+1)+C\,(s+3)^2}{(s+3)^2\,(s+1)}. \tag{92}$$

We need A,B,C such that

$$A\,(s+3)\,(s+1)+B\,(s+1)+C\,(s+3)^2=5\,s^2+34\,s+53. \tag{93}$$

Setting $s = -3$, we have

$$B(-3+1) = 45 - 102 + 53 = -4 \Longrightarrow B = 2. \tag{94}$$

Setting $s = -1$, we have

$$C(-1+3)^2 = 5 - 34 + 53 = 24 \Longrightarrow C = 6. \tag{95}$$

To determine A, we pick $s = 0$ to obtain

$$3A + B + 9C = 53 \Longrightarrow A = -1. \tag{96}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{5s^2+34s+53}{(s+3)^2(s+1)}\right\} &= \mathcal{L}^{-1}\left\{(-1)\frac{1}{s+3} + 2\frac{1}{(s+3)^2} + 6\frac{1}{s+1}\right\} \\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= -e^{-3t} + 2te^{-3t} + 6e^{-t}. \end{aligned} \tag{97}$$

Remark 25. As we have seen, when there are repeated roots, some of the unknowns cannot be determined easily. To determine these constants, we can either pick arbitrary values of s, or use the following idea. Let's say we need to determine A, B, C in

$$f(s) = A + B(s+1) + C(s+1)^2. \tag{98}$$

Setting $s = -1$ we have

$$A = f(-1). \tag{99}$$

To determine B, C we can either pick two arbitrary values of s, say $s = 0, 2$ and solve

$$A + B + C = f(0) \tag{100}$$

$$A + 3B + 9C = f(2) \tag{101}$$

or we can differentiate the equation:

$$f'(s) = B + C(s+1) \Longrightarrow B = f'(-1), \tag{102}$$

$$f''(s) = C \Longrightarrow C = f''(-1). \tag{103}$$

Is this "differentiation" method a simpler approach than setting arbitrary values of s? I guess it differs from case to case, and from person to person.

3. Some r_i's are complex.

   Let's say $r_i = \alpha + i\beta$, with multiplicity m. It can be shown that there must be $r_j = \alpha - i\beta$ with the same multiplicity. Then corresponding to r_i, r_j we introduce

$$\frac{C_1 s + D_1}{(s-\alpha)^2 + \beta^2} + \frac{C_2 s + D_2}{\left[(s-\alpha)^2 + \beta^2\right]^2} + \cdots + \frac{C_m s + D_m}{\left[(s-\alpha)^2 + \beta^2\right]^m}. \tag{104}$$

Equivalently, we can factorize Q into first order factors $s - r_i$ and second order factors $s^2 + ps + q$ where $s^2 + ps + q = 0$ would give conjugate complex roots. Then suppose there is a factor of $\left(s^2 + ps + q\right)^m$, the corresponding partial fractions are

$$\frac{C_1 s + D_1}{s^2 + ps + q} + \cdots + \frac{C_m s + D_m}{(s^2 + ps + q)^m}. \tag{105}$$

Example 26. Compute

$$\mathcal{L}^{-1}\left\{\frac{7s^2 + 23s + 30}{(s-2)(s^2 + 2s + 5)}\right\}. \tag{106}$$

Solution. Again, the degree of the nominator is lower. Check.

We write

$$\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}=\frac{A}{s-2}+\frac{B\,s+C}{s^2+2\,s+5}=\frac{A\left(s^2+2\,s+5\right)+(B\,s+C)\,(s-2)}{(s-2)\,(s^2+2\,s+5)}. \tag{107}$$

We need to find A, B, C such that

$$A\left(s^2+2\,s+5\right)+(B\,s+C)\,(s-2)=7\,s^2+23\,s+30. \tag{108}$$

Setting $s=2$ we have

$$A\,(4+4+5)=28+46+30=104\Longrightarrow A=8. \tag{109}$$

To find B, C, we need to set s to values different from 2 and obtain equations for B, C. There is a minor trick here that can make the equations simple. We notice that the B disappears if we set $s=0$. Setting $s=0$ we have

$$5\,A-2\,C=30\Longrightarrow C=5. \tag{110}$$

Finally comparing the s^2 terms (or setting s to yet another value) we have

$$A+B=7\Longrightarrow B=-1. \tag{111}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{8}{s-2}\right\}+\mathcal{L}^{-1}\left\{\frac{-s+5}{s^2+2\,s+5}\right\}. \tag{112}$$

The first term leads to $8\,e^{2t}$. To compute the second term, we further simplify

$$\begin{aligned}\frac{-s+5}{s^2+2\,s+5} &= -\frac{s-5}{(s+1)^2+4}\\ &= -\frac{s+1}{(s+1)^2+4}+3\,\frac{2}{(s+1)^2+2^2}.\end{aligned} \tag{113}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{-s+5}{s^2+2\,s+5}\right\} = -\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}+3\,\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \tag{114}$$

$$= -e^{-t}\cos 2\,t+3\,e^{-t}\sin 2\,t. \tag{115}$$

Summarizing, we have

$$\mathcal{L}^{-1}\left\{\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}\right\}=8\,e^{2t}-e^{-t}\cos 2\,t+3\,e^{-t}\sin 2\,t. \tag{116}$$

Remark 27. We summarize the method of partial fractions. The method represents a complicated ratio P/Q by a sum of simple ratios in which only simple polynomials of degrees no more than 2 are involved through the following procedure.

1. Factorize Q:

$$Q(s)=(s-r_1)\cdots(s-r_n). \tag{117}$$

2. Go through $r_1, \ldots, r_n$ and write down the terms of the RHS sum of

$$\frac{P}{Q}=\sum\cdots \tag{118}$$

according to the following rules:

i. If r_i is a single real root, write down

$$\frac{A_i}{s-r_i}. \tag{119}$$

ii. If r_i is a repeated real root, say with multiplicity m, write down

$$\frac{A_{i1}}{s-r_i}+\frac{A_{i2}}{(s-r_i)^2}+\cdots+\frac{A_{im}}{(s-r_i)^m}. \tag{120}$$

After this, discard those other copies of r_i from the list r_1, ..., r_n and move on to the next root. Note that the previous "single root" case is actually contained in this case.

iii. If $r_i = \alpha + i\,\beta$ is complex root with multiplicity m, then there must be another $r_j = \alpha - i\,\beta$ with the same multiplicity. Write down

$$\frac{C_{i1}\,s + D_{i1}}{(s-\alpha)^2 + \beta^2} + \cdots + \frac{C_{im}\,s + D_{im}}{\left[(s-\alpha)^2 + \beta^2\right]^m}. \tag{121}$$

For example, if

$$Q(s) = (s-1)(s-3)^3\,(s+i)\,(s-i), \tag{122}$$

we have six roots (counting multiplicity) $1, 3, 3, 3, -i, i$. Now to form the RHS, we go through this list one by one:

$$\begin{aligned}
1\text{: Single real root} &\implies \frac{A}{s-1}; && (123)\\
3\text{: repeated real root with multiplicity 3} &\implies \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3}; && (124)\\
\text{Ignore the remaining two 3's.} & && (125)\\
-i\text{: Complex root with multiplicity 1} &\implies \frac{E\,s+F}{s^2+1}; && (126)\\
\text{Ignore the complex conjugate } i. & && (127)
\end{aligned}$$

3. Determine the constants using the following procedure: We use the above example

$$Q(s) = (s-1)(s-3)^3\,(s+i)\,(s-i), \tag{128}$$

which gives

$$\frac{P}{Q} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3} + \frac{E\,s+F}{s^2+1} \tag{129}$$

leading to

$$P(s) = A\,(s-3)^3\,(s^2+1) + B\,(s-1)\,(s-3)^2\,(s^2+1) + C\,(s-1)\,(s-3)\,(s^2+1) + D\,(s-1)\,(s^2+1) + (E\,s+F)\,(s-1)\,(s-3)^3. \tag{130}$$

i. Set s to be each of the single real roots. This would immediately give all the constants corresponding to those single roots.

In our example, we see that setting $s=1$ immediately gives A.

ii. Set s to be the repeated real roots. This would immediately give all the constants in the last terms of the terms corresponding to those repeated roots.

In our example, setting $s=3$ immediately gives D.

- At this stage, you may want to try the "differentiation method". In our example, differentiating once we obtain

$$\begin{aligned}
P'(s) \;=\;& A\left[2\,(s-3)\,(s^2+1) + (s-3)^2\,(2\,s)\right]\\
&+ B\left[(s-3)^2\,(s^2+1) + 2\,(s-1)\,(s-3)\,(s^2+1) + 2\,s\,(s-1)\,(s-3)^2\right]\\
&+ C\left[(s-3)\,(s^2+1) + (s-1)\,(s^2+1) + 2\,s\,(s-1)\,(s-3)\right]\\
&+ D\left[s^2+1+2\,s\,(s-1)\right]\\
&+ E\left[(s-1)\,(s-3)^3\right] + (E\,s+F)\left[(s-3)^3 + 3\,(s-1)\,(s-3)^2\right].
\end{aligned} \tag{131}$$

Looks very complicated, but as soon as we substitute $s=3$, only C and D remain. As we have already found D, determining C is easy.

Differentiate again and then set $s=3$, we obtain one equation for B, C, D. Since we already know C, D, B is immediately determined.

iii. Set $s=0$.

iv. If there are still some constants need to be determined, compare the coefficient for the highest power term s^n of the RHS. Note that as P has lower degree, we always have $0 = \cdots$. In our example,

$$P(s) = A\,(s-3)^3\,(s^2+1) + B\,(s-1)\,(s-3)^2\,(s^2+1) + C\,(s-1)\,(s-3)\,(s^2+1) + D\,(s-1)\,(s^2+1) + (E\,s+F)\,(s-1)\,(s-3)^3. \tag{132}$$

The higher order term on the RHS is s^5. Assuming

$$P(s) = p_5\, s^5 + \cdots \tag{133}$$

we have

$$p_5 = A + B + E. \tag{134}$$

Note that this is equivalent to setting $s = \infty$.

v. Let's say there are k constants still need to be determined. Set s to be k arbitrary values. You will obtain k equations for these k costants, solve them.

In our example, $k=0$ if we have used the "differentiation method", $k=2$ if we haven't.

When F is not of the form P/Q, sometimes it can be transformed into that form.

Example 28. Compute

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s+2}{s-5}\right)\right\}. \tag{135}$$

Solution. We notice that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\ln\left(\frac{s+2}{s-5}\right)\right] = \frac{1}{s+2} - \frac{1}{s-5} \tag{136}$$

is of the form we just discussed.

Thus we have

$$\begin{aligned} -t\,\mathcal{L}^{-1}\left\{\ln\left(\frac{s+2}{s-5}\right)\right\} &= \mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[\ln\left(\frac{s+2}{s-5}\right)\right]\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} \\ &= e^{-2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - e^{5t}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= e^{-2t} - e^{5t}. \end{aligned} \tag{137}$$

Dividing both sides by $-t$ we reach

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s+2}{s-5}\right)\right\} = \left(e^{5t} - e^{-2t}\right)/t. \tag{138}$$

Remark 29. In the above we invert Laplace transforms via comparison with the Laplace transform table and some clever tricks. Actually there is a formula obtaining f from F. But this formula is beyond the level of this course, it reads:

$$\mathcal{L}^{-1}\{F\} = \frac{1}{2\,\pi\,i}\lim_{T\longrightarrow\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\,F(s)\,\mathrm{d}s \tag{139}$$

where γ is such that the contour is inside the region of convergence. Anyone who wants to perfect his/her contour integral/residue calculation techniques should try to compute all the above inverse transforms directly.

2. Solving initial value problems.

Now we have everything we need to solve linear equations. The following steps are followed:

1. Transform the equation.

   a. Transform the LHS;

b. Transform the RHS.

2. Solve the transformed equation to obtain $Y(s) = \mathcal{L}\{y\}$.
3. Compute $y = \mathcal{L}^{-1}\{Y\}$.

2.1. Linear equations with constant coefficient.

Linear equations with constant coefficients are transformed to algebraic equations.

Example 30. Solve

$$y'' - 2\,y' + 5\,y = 0, \qquad y(0) = 2, \quad y'(0) = 4 \tag{140}$$

using Laplace transform.

Solution. We follow the three steps.

1. Transform the equation.

   a. Transform the LHS. Denote $Y = \mathcal{L}\{y\}$.

      We compute

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y - 2\,s - 4. \tag{141}$$

$$\mathcal{L}\{y'\} = s\,Y - y(0) = s\,Y - 2. \tag{142}$$

   So

$$\mathcal{L}\{y'' - 2\,y' + 5\,y\} = s^2\,Y - 2\,s - 4 - 2\,s\,Y + 8 + 5\,Y = \left(s^2 - 2\,s + 5\right)Y - 2\,s. \tag{143}$$

   b. Transform the RHS.

$$\mathcal{L}\{0\} = 0. \tag{144}$$

2. Solve the transformed equation.

   The transformed equation reads

$$\left(s^2 - 2\,s + 5\right)Y - 2\,s + 4 = 0 \Longrightarrow Y = \frac{2\,s}{s^2 - 2\,s + 5}. \tag{145}$$

3. Compute y. We have

$$\begin{aligned} y &= \mathcal{L}^{-1}\{Y\} \\ &= 2\,\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2 + 2^2}\right\} \\ &= 2\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 2^2} + \frac{1}{2}\,\frac{2}{(s-1)^2 + 2^2}\right\} \\ &= 2\,e^t \cos 2\,t + e^t \sin 2\,t. \end{aligned} \tag{146}$$

Example 31. Solve

$$w'' + w = t^2 + 2, \qquad w(0) = 1, \quad w'(0) = -1 \tag{147}$$

using Laplace transform.

Solution. We follow the three steps.

1. Transform the equation.

   a. Transform the LHS. Denote $W = \mathcal{L}\{w\}$.

      We compute

$$\mathcal{L}\{w''\} = s^2\,W - s\,w(0) - w'(0) = s^2\,W - s + 1. \tag{148}$$

   This gives

$$\mathcal{L}\{w'' + w\} = s^2\,W - s + 1 + W = \left(s^2 + 1\right)W - s + 1. \tag{149}$$

   b. Transform the RHS. We have

$$\mathcal{L}\left\{t^2 + 2\right\} = \mathcal{L}\left\{t^2\right\} + 2\,\mathcal{L}\{1\} = \frac{2}{s^3} + \frac{2}{s}. \tag{150}$$

2. Solve the transformed equation.

   The transformed equation reads[3]

$$\left(s^2+1\right)W - s + 1 = \frac{2}{s^3} + \frac{2}{s} \Longrightarrow W = \frac{2\,s^2+2}{s^3\left(s^2+1\right)} + \frac{s-1}{s^2+1} = \frac{2}{s^3} + \frac{s-1}{s^2+1}. \tag{152}$$

3. Compute $w = \mathcal{L}^{-1}\{W\}$.

   We have

$$\mathcal{L}^{-1}\{W\} = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\}. \tag{153}$$

   We compute the RHS term by term.

   - $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$.

     We have

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = t^2. \tag{154}$$

   - $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\}$.

     We have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \cos t - \sin t.\end{aligned} \tag{155}$$

   Thus

$$w = t^2 + \cos t - \sin t. \tag{156}$$

Example 32. Solve

$$y'' - y' - 2\,y = -8\cos t - 2\sin t, \qquad y(\pi/2) = 1, \quad y'(\pi/2) = 0. \tag{157}$$

Solution. For problems with initial values given at points different from 0, the very first thing to do is to introduce a new unknown function w so that $w(0) = y(\pi/2)$, $w'(0) = y'(\pi/2)$. Such w can easily be constructed to be

$$w(t) = y(t + \pi/2). \tag{158}$$

The equation for w is then

$$w'' - w' - 2\,w = -8\cos\left(t + \frac{\pi}{2}\right) - 2\sin\left(t + \frac{\pi}{2}\right) = 8\sin t - 2\cos t. \tag{159}$$

To solve

$$w'' - w' - 2\,w = 8\sin t - 2\cos t, \qquad w(0) = 1, \quad w'(0) = 0 \tag{160}$$

we follow the three steps.

1. Transform the equation.

   a. Transform the LHS;

      We compute

$$\mathcal{L}\{w''\} = s^2\,W - s\,w(0) - w'(0) = s^2\,W - s. \tag{161}$$

$$\mathcal{L}\{w'\} = s\,W - w(0) = s\,W - 1. \tag{162}$$

   Thus

$$\mathcal{L}\{w'' - w' - 2\,w\} = \left(s^2 - s - 2\right)W - s + 1. \tag{163}$$

---

3. Note that, if we do not simplify the RHS, we would have to compute

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3\left(s^2+1\right)} + \frac{2}{s\left(s^2+1\right)} + \frac{s-1}{s^2+1}\right\} \tag{151}$$

and very complicated partial fraction calculation would be involved!

b. Transform the RHS;

We have

$$\mathcal{L}\{8\sin t-2\cos t\}=\frac{8}{s^2+1}-\frac{2\,s}{s^2+1}. \tag{164}$$

2. Solve the transformed equation.

The transformed equation is

$$\left(s^2-s-2\right)W-s+1=\frac{8-2\,s}{s^2+1} \tag{165}$$

which gives

$$W=\frac{8-2\,s}{(s^2-s-2)\,(s^2+1)}+\frac{s-1}{s^2-s-2}. \tag{166}$$

3. Compute $w=\mathcal{L}^{-1}(W)$.

We need to compute

$$\mathcal{L}^{-1}\left\{\frac{8-2\,s}{(s^2-s-2)\,(s^2+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-s-2}\right\}. \tag{167}$$

- $\mathcal{L}^{-1}\left\{\frac{8-2\,s}{(s^2-s-2)\,(s^2+1)}\right\}$.

  We use the method of partial fractions. As $s^2-s-2=(s-2)\,(s+1)$, write

$$\frac{8-2\,s}{(s^2-s-2)\,(s^2+1)}=\frac{A}{s-2}+\frac{B}{s+1}+\frac{C\,s+D}{s^2+1}. \tag{168}$$

The RHS is

$$\frac{A\,(s+1)\left(s^2+1\right)+B\,(s-2)\left(s^2+1\right)+(C\,s+D)\,(s-2)\,(s+1)}{(s^2-s-2)\,(s^2+1)}. \tag{169}$$

We need A—D such that

$$A\,(s+1)\left(s^2+1\right)+B\,(s-2)\left(s^2+1\right)+(C\,s+D)\,(s-2)\,(s+1)=8-2\,s. \tag{170}$$

Setting $s=2$ we obtain

$$15\,A=4\Longrightarrow A=4/15. \tag{171}$$

Setting $s=-1$ we reach

$$-6\,B=10\Longrightarrow B=-5/3. \tag{172}$$

Setting $s=0$, we have

$$A-2\,B-2\,D=8\Longrightarrow D=-11/5. \tag{173}$$

Finally, comparing the coefficients of s^3 on both sides, we have

$$A+B+C=0\Longrightarrow C=7/5. \tag{174}$$

Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{8-2\,s}{(s^2-s-2)\,(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{4/15}{s-2}+\frac{-5/3}{s+1}+\frac{\frac{7}{5}\,s-\frac{11}{5}}{s^2+1}\right\}\\ &= \frac{4}{15}\,e^{2t}-\frac{5}{3}\,e^{-t}+\frac{7}{5}\cos t-\frac{11}{5}\sin t.\end{aligned} \tag{175}$$

- $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-s-2}\right\}$.

  We apply the method of partial fractions again. Write

$$\frac{s-1}{s^2-s-2}=\frac{A}{s-2}+\frac{B}{s+1}=\frac{A\,(s+1)+B\,(s-2)}{s^2-s-2}. \tag{176}$$

We easily obtain

$$A=1/3,\quad B=2/3. \tag{177}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-s-2}\right\}=\mathcal{L}^{-1}\left\{\frac{1/3}{s-2}\right\}+\mathcal{L}^{-1}\left\{\frac{2/3}{s+1}\right\}=\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}. \tag{178}$$

Putting things together, we have

$$\begin{aligned} w &= \mathcal{L}^{-1}\left\{\frac{8-2s}{(s^2-s-2)(s^2+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-s-2}\right\} \\ &= \frac{4}{15}e^{2t}-\frac{5}{3}e^{-t}+\frac{7}{5}\cos t-\frac{11}{5}\sin t+\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t} \\ &= \frac{3}{5}e^{2t}-e^{-t}+\frac{7}{5}\cos t-\frac{11}{5}\sin t. \end{aligned} \tag{179}$$

Finally, we return to y:

$$w(t)=y(t+\pi/2)\Longrightarrow y(t)=w(t-\pi/2)=\frac{3}{5}e^{2t-\pi}-e^{\frac{\pi}{2}-t}+\frac{7}{5}\sin t+\frac{11}{5}\cos t. \tag{180}$$

2.2. Solving higher order equations.

There is no difficulty generalizing this method to higher order constant-coefficient equations.

Example 33. Solve

$$y'''-y''+y'-y=0, \qquad y(0)=1, \quad y'(0)=1, \quad y''(0)=3. \tag{181}$$

Solution. The "three steps" are still the same.

1. Transform the equation.

   a. Transform the LHS;

      We have

$$\mathcal{L}\{y'''\}=s^3Y-s^2y(0)-sy'(0)-y''(0)=s^3Y-s^2-s-3; \tag{182}$$

$$\mathcal{L}\{y''\}=s^2Y-sy(0)-y'(0)=s^2Y-s-1; \tag{183}$$

$$\mathcal{L}\{y'\}=sY-y(0)=sY-1. \tag{184}$$

   Overall

$$\mathcal{L}\{y'''-y''+y'-y\}=\left(s^3-s^2+s-1\right)Y-s^2-3. \tag{185}$$

   b. Transform the RHS;

      We have

$$\mathcal{L}\{0\}=0. \tag{186}$$

2. Solve the transformed equation.

   The transformed equation reads

$$\left(s^3-s^2+s-1\right)Y-s^2-3=0\Longrightarrow Y=\frac{s^2+3}{(s^3-s^2+s-1)}=\frac{s^3+3}{(s-1)(s^2+1)}. \tag{187}$$

3. Invert $y=\mathcal{L}^{-1}(Y)$.

   We need to compute

$$\mathcal{L}^{-1}\left\{\frac{s^2+3}{(s-1)(s^2+1)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s^2+1)}\right\}. \tag{188}$$

For the first term we simply compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}=e^t. \tag{189}$$

For the second term we write

$$\frac{2}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}=\frac{A\left(s^2+1\right)+(Bs+C)(s-1)}{(s-1)(s^2+1)}. \tag{190}$$

We need A, B, C such that

$$A\left(s^2+1\right)+(B\,s+C)\,(s-1)=2. \tag{191}$$

Setting $s=1$ we reach

$$A=1. \tag{192}$$

Next we set $s=0$ to obtain

$$A-C=2\Longrightarrow C=-1. \tag{193}$$

Finally comparing the coefficients of s^2:

$$A+B=0\Longrightarrow B=-1. \tag{194}$$

Summarize:

$$A=1, \qquad B=-1, \qquad C=-1. \tag{195}$$

It is clear (through substituting back into the equations for A, B, C) that these are correct. Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)\,(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}\\ &= e^t-\cos t-\sin t.\end{aligned} \tag{196}$$

Summarizing, we have

$$y=2\,e^t-\cos t-\sin t. \tag{197}$$

2.3. Linear equations with variable coefficient.

When the coefficients are not constants, the method of Laplace transform becomes somewhat awkward. Nevertheless, it is still useful when the coefficients are really simple.[4]

Example. Solve

$$y''+3\,t\,y'-6\,y=1, \qquad y(0)=0, \quad y'(0)=0. \tag{199}$$

Solution. The three steps are still the same. However the details inside each step is different from the constant-coefficient case.

1. Transform the equation. As before, we denote $Y=\mathcal{L}\{y\}$.

   a. Transform the LHS;

      We compute

$$\mathcal{L}\{y''\}=s^2\,Y-s\,y(0)-y'(0)=s^2\,Y. \tag{200}$$

   To compute $\mathcal{L}\{t\,y'\}$, recall that

$$\mathcal{L}\{t^n\,f\}=(-1)^n\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f\}. \tag{201}$$

   Thus

$$\mathcal{L}\{t\,y'\}=-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{y'\}=-\frac{\mathrm{d}}{\mathrm{d}s}[s\,Y-y(0)]=-\frac{\mathrm{d}}{\mathrm{d}s}(s\,Y)=-Y-s\,\frac{\mathrm{d}Y}{\mathrm{d}s}. \tag{202}$$

   Thus

$$\mathcal{L}\{y''+3\,t\,y'-6\,y\}=s^2\,Y-3\,Y-3\,s\,\frac{\mathrm{d}Y}{\mathrm{d}s}-6\,Y=\left(s^2-9\right)Y-3\,s\,\frac{\mathrm{d}Y}{\mathrm{d}s}. \tag{203}$$

   b. Transform the RHS. We have

$$\mathcal{L}\{1\}=\frac{1}{s}. \tag{204}$$

---

4. Recall that, the only variable coefficient equation we can solve (systematically and completely) so far is the Cauchy-Euler equation

$$a\,t^2\,y''+b\,t\,y'+c\,y=0. \tag{198}$$

As we will soon see, the method of Laplace transform can solve some variable coefficient equations that are not Cauchy-Euler. So the method of Laplace transform indeed strengthens our ability of solving variable coefficient equations.

On the other hand, if we try to use Laplace transform to solve a Cauchy-Euler equation, we will obtain another Cauchy-Euler equation (with variable s). So Laplace transform doesn't help in the case of Cauchy-Euler equations.

2. Solve the transformed equation.

   The transformed equation reads

$$-3\,s\,\frac{\mathrm{d}Y}{\mathrm{d}s}+\left(s^2-9\right)Y=\frac{1}{s}. \tag{205}$$

   Divide both sides by $-3\,s$, we reach

$$\frac{\mathrm{d}Y}{\mathrm{d}s}+\left(\frac{3}{s}-\frac{s}{3}\right)Y=-\frac{1}{3\,s^2}. \tag{206}$$

   Recall how to solve first order linear equations:

$$y'+Py=Q\Longrightarrow\left(e^{\int P}y\right)'=e^{\int P}Q\Longrightarrow y=e^{-\int P}\int e^{\int P}Q. \tag{207}$$

   The integration factor is

$$e^{\int\frac{3}{s}-\frac{s}{3}}=e^{-\frac{s^2}{6}}s^3. \tag{208}$$

   Note that in general the domain of Y is contained in $s>0$. Therefore $\ln|s|=\ln s$. We have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[s^3\,e^{-\frac{s^2}{6}}Y\right]=e^{-\frac{s^2}{6}}s^3\left(-\frac{1}{3\,s^2}\right)=-\frac{s}{3}\,e^{-\frac{s^2}{6}}. \tag{209}$$

   Integrating, we obtain

$$s^3\,e^{-\frac{s^2}{6}}Y=e^{-\frac{s^2}{6}}+C\Longrightarrow Y=\frac{1}{s^3}+C\,s^{-3}\,e^{\frac{s^2}{6}}. \tag{210}$$

   To determine the constant C, we use the following property:

$$\lim_{s\nearrow+\infty}\mathcal{L}\{f\}(s)=0 \tag{211}$$

   for all reasonable f.[5] Thus $Y(s)\longrightarrow 0$ as $s\longrightarrow+\infty$ which immediately leads to $C=0$. Thus

$$Y=\frac{1}{s^3}. \tag{212}$$

3. Compute $y=\mathcal{L}^{-1}\{Y\}$.

   We compute

$$y=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{1}{2}\,t^2. \tag{213}$$

2.4. Solving linear systems.

There is no difficulty generalizing this method to linear systems.

Example 34. Solve

$$\begin{aligned} x' &= 3\,x-2\,y, \qquad x(0)=1; && (214)\\ y' &= 3\,y-2\,x; \qquad y(0)=1. && (215)\end{aligned}$$

Solution. Still the same old three steps. But first we write the system into

$$\begin{aligned} x'-3\,x+2\,y &= 0, \qquad x(0)=1 && (216)\\ y'-3\,y+2\,x &= 0, \qquad y(0)=1. && (217)\end{aligned}$$

We use X,Y to denote the Laplace transforms of x,y.

1. Transform the equations.

   a. Transform the LHS;

      We have

$$\mathcal{L}\{x'\}=s\,X-x(0)=s\,X-1 \tag{218}$$

$$\mathcal{L}\{y'\}=s\,Y-y(0)=s\,Y-1. \tag{219}$$

5. More specifically, for f piecewise continuous and of exponential order.

Thus the LHS are

$$(s-3)\,X+2\,Y-1 \tag{220}$$

and

$$(s-3)\,Y+2\,X-1. \tag{221}$$

b. Transform the RHS;

The transformed RHS are simply $0,0$.

2. Solve the transformed system.

   The transformed system is

$$\begin{aligned}(s-3)\,X+2\,Y &= 1 \qquad (222)\\ 2\,X+(s-3)\,Y &= 1. \qquad (223)\end{aligned}$$

Solving it[6] we obtain

$$X=\frac{s-5}{(s-3)^2-4}=\frac{1}{s-1}; \qquad Y=\frac{1}{s-1}. \tag{224}$$

3. Invert $x=\mathcal{L}^{-1}(X)$, $y=\mathcal{L}^{-1}(Y)$.

   Clearly we have

$$x=y=e^t. \tag{225}$$

3. Convolution, transfer function, impulse response function.

From the above we see that solving constant-coefficient ODEs consists of the following steps:

1. Transform the equation (LHS and RHS).
2. Solve the transformed equation.
3. Transform back.

In practice, we often face the following situation: We need to compute the output of a system for many different inputs, sometimes we also need to find out a general relation between the inputs and the outputs.

After mathematical modeling, this situation is translated to the following. Given a differential operator (corresponding to the system), we need to compute the solutions for many different right hand sides/initial values. For example, we may need to consider problems like

$$y''+y=g(t); \qquad y(0)=a, \quad y'(0)=b \tag{226}$$

and need to solve it for many different g's.

We follow our three steps.

1. Transform the equation. Using the initial values and denoting $Y(s)=\mathcal{L}\{y\}$, $G(s)=\mathcal{L}\{g\}$, we have

$$\left(s^2+1\right)Y=G+s\,a+b. \tag{227}$$

2. Solve the transformed equation. We have

$$Y(s)=\frac{1}{s^2+1}\,[G(s)+s\,a+b]. \tag{228}$$

3. Transform back. We need to compute

$$y=\mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{H(s)\,\tilde{G}(s)\right\} \tag{229}$$

   where $H(s):=\frac{1}{s^2+1}$, and $\tilde{G}(s):=G(s)+s\,a+b$.

6. Either using Cramer's rule, or multiply the first equation by 2 and the second by $(s-3)$, then take the difference to obtain Y, etc.

From the above we can make the following observations.

A) The Laplace transform of the solution is the product of two functions. One of them, $H(s)$, is determined by the differential operator (the system); The other, $\tilde{G}(s)$, is only dependent on the right hand side and the initial values (the inputs).

   Thus it is possible to simplify the process of solving equations with many different inputs, as $H(s)$ remains the same.

B) To be able to take advantage of the above observation, we need to be able to compute the inverse Laplace transforms of product of two functions. Or more specifically, if we know that $\mathcal{L}^{-1}\{H\} = h$, $\mathcal{L}^{-1}\left\{\tilde{G}\right\} = \tilde{g}$, can we obtain $\mathcal{L}^{-1}\left\{H\tilde{G}\right\}$ using h and $\tilde{g}$?

   The answer is yes:

$$\mathcal{L}^{-1}\left\{H\tilde{G}\right\} = h * \tilde{g}, \tag{230}$$

   the **convolution** of h and $\tilde{g}$.

Definition 35. (Transfer function/Impulse response function) *Consider the linear system*

$$a\,y'' + b\,y' + c\,y = g, \qquad y(0) = y'(0) = 0, \tag{231}$$

*Let $Y = \mathcal{L}\{y\}, G = \mathcal{L}\{g\}$. The function $H(s) = Y/G$ is called the **transfer function** of the linear system. Its inverse Laplace transform, $h(t) = \mathcal{L}^{-1}\{H(s)\}(t)$, is called the **impulse response function** for the system.*

Remark 36. Note that, $h(t)$ solve the equation

$$a\,y'' + b\,y' + c\,y = \delta(t) \tag{232}$$

where $\delta(t)$ is a special function such that

$$\mathcal{L}\{\delta\} = 1. \tag{233}$$

It turns out that this $\delta(t)$ cannot be a usual function. It belongs to the so-called "generalized functions".

Example 37. Find the transfer function and the impulse response function for

$$y'' + 9\,y = g(t). \tag{234}$$

Solution. Taking Laplace transform we have (recall that when talking about transfer function, we always assume $y(0) = y'(0) = 0$.)

$$s^2\,Y + 9\,Y = G \Longrightarrow H = \frac{Y}{G} = \frac{1}{s^2 + 9}. \tag{235}$$

So the transfer function is

$$H(s) = \frac{1}{s^2 + 9}. \tag{236}$$

The impulse response function is then given by

$$h(t) = \mathcal{L}^{-1}\{H\} = \frac{1}{3}\sin 3\,t. \tag{237}$$

Definition 38. (Convolution) *Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$, denoted $f * g$, is defined by*

$$(f * g)(t) := \int_0^t f(t - v)\,g(v)\,\mathrm{d}v. \tag{238}$$

Example 39. For example, the convolution of $f = t$ and $g = t^2$ is

$$t * t^2 = \int_0^t (t - v)\,v^2\,\mathrm{d}v = \int_0^t t\,v^2 - v^3\,\mathrm{d}v = \frac{1}{3}\,t^4 - \frac{1}{4}\,t^4 = \frac{t^4}{12}. \tag{239}$$

Recalling our purpose of introducing convolution, we should prove the following theorem.

Theorem 40. (Convolution Theorem) *Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then*

$$\mathcal{L}\{f * g\}(s) = F(s)\, G(s), \tag{240}$$

or equivalently

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = (f * g)(t). \tag{241}$$

Proof. We use definition.

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^\infty e^{-st} \int_0^t f(t-v)\, g(v)\, \mathrm{d}v\, \mathrm{d}t \\ &= \iint_{0<v<t} e^{-st} f(t-v)\, g(v)\, \mathrm{d}v\, \mathrm{d}t \\ &= \int_0^\infty \left[\int_v^\infty e^{-st} f(t-v)\, g(v)\, \mathrm{d}t \right] \mathrm{d}v \\ &= \int_0^\infty g(v) \left[\int_v^\infty e^{-st} f(t-v)\, \mathrm{d}t \right] \mathrm{d}v. \end{aligned} \tag{242}$$

To evaluate the inner integral, we set $x = t - v$. Then $\mathrm{d}t = \mathrm{d}x$, $e^{-st} = e^{-sx} e^{-sv}$. Thus

$$\int_v^\infty e^{-st} f(t-v)\, \mathrm{d}t = e^{-sv} \int_0^\infty e^{-sx} f(x)\, \mathrm{d}x = e^{-sv} F(s). \tag{243}$$

Substituting into the original integral, we have

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty g(v) \left[\int_v^\infty e^{-st} f(t-v)\, \mathrm{d}t \right] \mathrm{d}v = F(s) \int_0^\infty g(v)\, e^{-sv}\, \mathrm{d}v = F(s)\, G(s). \tag{244}$$

Thus ends the proof. □

Example 41. Use the convolution theorem to obtain a formula for the solution to the given initial value problem.

$$y'' - 2\, y' + y = g(t); \qquad y(0) = -1, \quad y'(0) = 1. \tag{245}$$

Solution. Using

$$\mathcal{L}\{y''\} = s^2\, Y - s\, y(0) - y'(0) = s^2\, Y + s - 1, \tag{246}$$

$$\mathcal{L}\{y'\} = s\, Y - y(0) = s\, Y + 1, \tag{247}$$

we have the transformed equation

$$\left(s^2 - 2\, s + 1\right) Y = G(s) - s + 3 \Longrightarrow Y(s) = \frac{1}{(s-1)^2}\, G(s) - \frac{1}{(s-1)} + \frac{2}{(s-1)^2}. \tag{248}$$

Transforming back, we have

$$\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^2}\, G(s) \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^2} \right\} * g(t) = \left(e^t\, t\right) * g(t); \tag{249}$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\} = e^t; \tag{250}$$

$$\mathcal{L}^{-1}\left\{ \frac{2}{(s-1)^2} \right\} = 2\, e^t\, t. \tag{251}$$

So the formula for the solution reads

$$y(t) = 2\, t\, e^t - e^t + \left(e^t\, t\right) * g(t) = 2\, t\, e^t - e^t + \int_0^t e^{t-v}\, (t-v)\, g(v)\, \mathrm{d}v. \tag{252}$$

Example 42. Compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+1)}\right\} \tag{253}$$

Solution. Using Convolution Theorem, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= 1*\sin t\\ &= \int_0^t \sin v\,\mathrm{d}v\\ &= 1-\cos t. \end{aligned}\tag{254}$$

Remark 43. Recall that, without Convolution Theorem, we need to use the method of partial fractions: Write

$$\frac{1}{s\,(s^2+1)} = \frac{A}{s} + \frac{B\,s+C}{s^2+1}, \tag{255}$$

determine A, B, C, and so on. Clearly our current approach (using Convolution Theorem) is better.

On the other hand, keep in mind that in many cases, it is not an easy task to compute convolutions.

Example 44. Find the Laplace transform of

$$f(t) = \int_0^t (t-v)\,e^{3v}\,\mathrm{d}v. \tag{256}$$

Solution. We recognize that in fact

$$f(t) = t * e^{3t}. \tag{257}$$

Thus

$$\mathcal{L}\{f\} = \mathcal{L}\{t\}\,\mathcal{L}\{e^{3t}\} = \frac{1}{s^2}\,\frac{1}{s-3} = \frac{1}{s^2\,(s-3)}.$$

Example 45. Compute

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\}. \tag{258}$$

Solution. We have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= [\cos t+\sin t]*\sin t\\ &= \int_0^t \cos(t-v)\sin v\,\mathrm{d}v + \int_0^t \sin(t-v)\sin v\,\mathrm{d}v\\ &= \int_0^t \frac{1}{2}[\sin t-\sin(t-2\,v)]\,\mathrm{d}v + \int_0^t \frac{1}{2}[\cos(t-2\,v)-\cos t]\,\mathrm{d}v\\ &= \frac{1}{2}\,t\,(\sin t-\cos t) - \frac{1}{2}\left[\int_0^t \sin(t-2\,v)\,\mathrm{d}v + \int_0^t \cos(t-2\,v)\,\mathrm{d}v\right]\\ &= \frac{1}{2}\,t\,(\sin t-\cos t) - \frac{1}{4}\left[\cos(t-2\,v)\,|_{v=0}^{v=t} - \sin(t-2\,v)\,|_0^t\,\right]\\ &= \frac{1}{2}\,t\,(\sin t-\cos t) - \frac{1}{2}\sin t. \end{aligned}\tag{259}$$

Theorem 46. (Other properties of convolution) *Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0,\infty)$. Then*

a) $f*g = g*f$;

b) $f*(g+h) = (f*g)+(f*h)$;

c) $(f*g)*h = f*(g*h)$;

d) $f*0=0$.

Proof. One way to prove these properties is to use definition (p.425 in the textbook). Here we use the Convolution Theorem. We use F, G, H to denote the Laplace transforms of f, g, h.

a) We have

$$f*g = \mathcal{L}^{-1}\{\mathcal{L}\{f*g\}\} = \mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{G\,F\} = g*f. \tag{260}$$

b) We have

$$\begin{aligned} f*(g+h) &= \mathcal{L}^{-1}\{\mathcal{L}\{f*(g+h)\}\} \\ &= \mathcal{L}^{-1}\{F\,(G+H)\\ &= \mathcal{L}^{-1}\{FG+FH\} \\ &= \mathcal{L}^{-1}\{FG\}+\mathcal{L}^{-1}\{FH\} \\ &= f*g+f*h. \end{aligned} \tag{261}$$

c) We have

$$\begin{aligned} (f*g)*h &= \mathcal{L}^{-1}\{\mathcal{L}\{(f*g)*h\}\} \\ &= \mathcal{L}^{-1}\{\mathcal{L}\{f*g\}\,H\} \\ &= \mathcal{L}^{-1}\{(FG)\,H\} \\ &= \mathcal{L}^{-1}\{F\,(G\,H)\} \\ &= f*\mathcal{L}^{-1}\{G\,H\} \\ &= f*(g*h). \end{aligned} \tag{262}$$

d) is obvious.

□

4. Integral-differential equation.

Using Convolution Theorem, we can solve a class of equations involving both derivatives and integrals.

Example 47. Solve

$$y(t) + 3\int_0^t y(v)\sin(t-v)\,\mathrm{d}v = t. \tag{263}$$

Solution. We try following the same three steps.

1. Transform the equation. The key is to notice that the integral term is in fact a convolution:

$$\int_0^t y(v)\sin(t-v)\,\mathrm{d}v = y(t)*\sin t. \tag{264}$$

Thus taking Laplace transform of both sides, we have

$$Y(s) + 3\,Y(s)\,\frac{1}{1+s^2} = \frac{1}{s^2}. \tag{265}$$

2. Solve the transformed equation. We have

$$Y(s) = \frac{1+s^2}{(4+s^2)\,s^2} = \frac{1}{s^2} - 3\,\frac{1}{(4+s^2)\,s^2} = \frac{1}{4}\,\frac{1}{s^2} - \frac{3}{4}\,\frac{1}{s^2+4}. \tag{266}$$

3. Transform back. Compute

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{4}\,\frac{1}{s^2} - \frac{3}{4}\,\frac{1}{s^2+4}\right\} = \frac{t}{4} - \frac{3}{8}\sin 2\,t. \tag{267}$$

Example 48. Solve

$$y'(t)+y(t)-\int_0^t y(v)\sin(t-v)\,\mathrm{d}v=-\sin t,\qquad y(0)=1. \tag{268}$$

Solution. Again,

1. Transform the equation. We have

$$\mathcal{L}\{y'\}=s\,Y-y(0)=s\,Y-1. \tag{269}$$

and

$$\mathcal{L}\left\{\int_0^t y(v)\sin(t-v)\,\mathrm{d}v\right\}=Y(s)\,\mathcal{L}\{\sin t\}=\frac{Y(s)}{1+s^2}. \tag{270}$$

So the equation is transformed into

$$s\,Y+Y-1-\frac{Y(s)}{1+s^2}=-\frac{1}{1+s^2}. \tag{271}$$

2. Solve the transformed equation. We have

$$Y=\frac{s}{s^2+s+1}. \tag{272}$$

3. Transform back. Compute

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y\}\\ &= \mathcal{L}^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2}-\frac{1/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2}\right\}\\ &= e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}\,t}{2}-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}\,t}{2}\right]. \end{aligned} \tag{273}$$

5. Dealing with non-classical functions.

So far only continuous functions appear in our problems. However, in practice other types of functions may get involved. For example, consider an object under forcing.

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t}=F(t). \tag{274}$$

It may happen that

- The forcing suddenly changes. Say $F(t)$ is 0 for $t<1$, but starts to increase rapidly at $t=0$, and reaches 1 at $t=1.0001$. A good way to mathematically model such forcing is to write

$$F(t)=\begin{cases} 0 & 0<t<1\\ 1 & t>1\end{cases}. \tag{275}$$

  Such F is not continuous anymore, it has "jumps".

- The forcing is asserted in an impulsive way, say the object is hit by a hammer at $t=1$. Thus the force is 0 for $t<1$ and $t>1.0001$, but is very large between $t=1$ and $t=1.0001$. Furthermore, in such cases it is reasonable to assume that the "impulse":

$$\int_1^{1.0001} F(t)\,\mathrm{d}t \tag{276}$$

  is a fixed number, say 1.[7] A good way to model such forcing is to write

$$F(t)=\delta(t-1) \tag{277}$$

  where $\delta(t)$ is a "generalized function", satisfying

$$\delta(t)=0\text{ whenever }t\neq 0;\qquad \int_{-\varepsilon}^{\varepsilon}\delta(t)\,\mathrm{d}t=1\text{ for any }\varepsilon>0. \tag{278}$$

7. Recall the "impulse response function"!

Clearly, to study such systems, we need to be able to perform Laplace transform on the above types of functions. We study them one by one.

5.1. Functions with jump(s).

On first sight, the only way to compute Laplace transforms of functions with jump is to use definition. For example, consider

$$g(t)=\begin{cases}1 & 0<t<3\\ t & t>3\end{cases}. \tag{279}$$

By definition

$$\mathcal{L}\{g\}(s)=\int_0^\infty e^{-st}\,g(t)\,\mathrm{d}t=\int_0^3 e^{-st}\,\mathrm{d}t+\int_3^\infty t\,e^{-st}\,\mathrm{d}t. \tag{280}$$

We have to compute both integrals from scratch, despite the fact that we know both $\mathcal{L}\{1\}$ and $\mathcal{L}\{t\}$, that is $\int_0^\infty e^{-st}\,\mathrm{d}t$ and $\int_0^\infty t\,e^{-st}\,\mathrm{d}t$ – after all there is no relation between $\int_0^3 e^{-st}\,\mathrm{d}t$ and $\int_0^\infty e^{-st}$, or is there?

Indeed there is (in some sense). We can take advantage of our knowing $\mathcal{L}\{1\}$ and $\mathcal{L}\{t\}$ when computing $\mathcal{L}\{g\}$, through the use of the following "unit step function".

Definition 49. (Unit step function) *The* ***unit step function*** $u(t)$ *is defined by*

$$u(t):=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}. \tag{281}$$

From the definition we see that for any constant M,

$$M\,u(t-a)=\begin{cases}0 & t<a\\ M & t>a\end{cases}. \tag{282}$$

Now suppose we have a function f such that $f(a+)=f(a-)+M$, it is easy to see that $f-M\,u(t-a)$ is continuous at a. More generally, let us notice that the function

$$f(t)+g(t)\,u(t-a) \tag{283}$$

takes value $f(t)$ for $0<t<a$, and value $f(t)+g(t)$ for $t>a$. Similar results hold when there are three, four, or more terms in the sum.

In general, the representation of a function

$$g(t)=\begin{cases}g_1(t) & 0<t<t_1\\ \vdots & \\ g_k(t) & t_{k-1}<t<t_k\end{cases} \tag{284}$$

is

$$g(t)=g_1(t)+[g_2(t)-g_1(t)]\,u(t-t_1)+[g_3(t)-g_2(t)]\,u(t-t_2)+\cdots+[g_k(t)-g_{k-1}(t)]\,u(t-t_{k-1}). \tag{285}$$

One can check, using the definition of the unit step function u, that $g(t)$ indeed takes the correct values in each interval $[t_i,t_{i+1}]$.

Example 50. Express the given function using unit step functions.

$$g(t)=\begin{cases}0 & 0<t<1\\ 2 & 1<t<2\\ 1 & 2<t<3\\ 3 & 3<t\end{cases}. \tag{286}$$

Solution. We have $g_1(t)=0,\ g_2(t)=2,\ g_3(t)=1,\ g_4(t)=3$. Thus

$$g(t)=2\,u(t-1)-u(t-2)+2\,u(t-3).$$

Example 51. Express

$$g(t)=\begin{cases}0 & 0<t<2\\ t+1 & 2<t\end{cases} \tag{287}$$

using unit jump function.

Solution. We have

$$g(t) = (t+1)\,u(t-2). \tag{288}$$

Remember that we introduce the unit jump function to compute Laplace transforms of discontinuous functions. From the above, we see that we need to be able to compute

$$\mathcal{L}\{g(t)\,u(t-a)\}. \tag{289}$$

Using definition of Laplace transform, we have

$$\begin{aligned}
\mathcal{L}\{g(t)\,u(t-a)\} &= \int_0^\infty e^{-st}\,g(t)\,u(t-a)\,\mathrm{d}t \\
&= \int_a^\infty e^{-st}\,g(t)\,\mathrm{d}t \\
&= e^{-as}\int_a^\infty e^{-s(t-a)}\,g(t)\,\mathrm{d}t \\
&= e^{-as}\int_a^\infty e^{-sv}\,g(v+a)\,\mathrm{d}v \\
&= e^{-as}\,\mathcal{L}\{g(t+a)\}(s).
\end{aligned} \tag{290}$$

In particular, we have

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}. \tag{291}$$

Also, if we define $f(t)$ such that

$$f(t-a) = g(t), \tag{292}$$

then

$$g(t+a) = f(t) \tag{293}$$

which leads to

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,F(s). \tag{294}$$

The corresponding inverse relation is

$$\mathcal{L}^{-1}\left\{e^{-as}\,F(s)\right\} = f(t-a)\,u(t-a). \tag{295}$$

Remark 52. A quick summary. The common situations are

$$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}(s), \tag{296}$$

and

$$\mathcal{L}^{-1}\left\{e^{-as}\,F(s)\right\} = f(t-a)\,u(t-a). \tag{297}$$

Example 53. Compute the Laplace transform of

$$g(t) = \begin{cases} 0 & 0<t<1 \\ 2 & 1<t<2 \\ 1 & 2<t<3 \\ 3 & 3<t \end{cases}. \tag{298}$$

Solution. We have already found out that

$$g(t) = 2\,u(t-1) - u(t-2) + 2\,u(t-3). \tag{299}$$

Thus

$$\mathcal{L}\{g\}(s) = 2\,\mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} + 2\,\mathcal{L}\{u(t-3)\} = \frac{2\,e^{-s} - e^{-2s} + 2\,e^{3s}}{s}. \tag{300}$$

Example 54. Compute Laplace transform of

$$g(t) = \begin{cases} 0 & 0 < t < 2 \\ t+1 & 2 < t \end{cases} \tag{301}$$

Solution. We have already solved

$$g(t) = (t+1)\, u(t-2). \tag{302}$$

Let $\tilde{g}(t) = t+1$. We have

$$\mathcal{L}\{\tilde{g}(t)\, u(t-2)\} = e^{-2s}\, \mathcal{L}\{\tilde{g}(t+2)\} = e^{-2s}\, \mathcal{L}\{t+3\} = e^{-2s}\left[\frac{1}{s^2} + \frac{3}{s}\right]. \tag{303}$$

Example 55. Determine the inverse Laplace transform of

$$\frac{e^{-2s}}{s-1}. \tag{304}$$

Solution. We have

$$\mathcal{L}^{-1}\left\{e^{-as}\, F(s)\right\} = f(t-a)\, u(t-a). \tag{305}$$

Comparing with the problem, we have $a = 2$, and $F(s) = \frac{1}{s-1}$. Inverting $F(s)$ we have

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t. \tag{306}$$

Thus

$$f(t-2) = e^{t-2}. \tag{307}$$

So finally

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} = e^{t-2}\, u(t-2). \tag{308}$$

Example 56. Compute the inverse Laplace transform of

$$\frac{s\, e^{-3s}}{s^2 + 4\, s + 5}. \tag{309}$$

Solution. Comparing with

$$\mathcal{L}^{-1}\left\{e^{-as}\, F(s)\right\} = f(t-a)\, u(t-a). \tag{310}$$

we have $a = 3$, $F(s) = \frac{s}{s^2 + 4\, s + 5}$. We compute

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4\, s + 5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2 + 1} - \frac{2}{(s+2)^2 + 1}\right\} \\ &= e^{-2t}\, [\cos t - 2 \sin t]. \end{aligned} \tag{311}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s\, e^{-3s}}{s^2 + 4\, s + 5}\right\} &= f(t-3)\, u(t-3) \\ &= e^{-2(t-3)}\, [\cos(t-3) - 2 \sin(t-3)]\, u(t-3). \end{aligned} \tag{312}$$

Now we solve a few equations with discontinuous right hand sides.

Example 57. Solve and sketch the solution.

$$y'' + y = u(t-3); \qquad y(0) = 0, \quad y'(0) = 1. \tag{313}$$

Solution. We follow the same old three steps.

1. Transform the equation. Compute

$$\mathcal{L}\{y''\} = s^2\, Y - s\, y(0) - y'(0) = s^2\, Y - 1. \tag{314}$$

Thus the equation is transformed into

$$\left(s^2+1\right)Y=\frac{e^{-3s}}{s}+1. \tag{315}$$

2. Solve the transformed equation.

$$Y=\frac{e^{-3s}}{s\,(s^2+1)}+\frac{1}{s^2+1}. \tag{316}$$

3. Transform back. Compute

$$\begin{aligned} y=\mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s\,(s^2+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}(t-3)\,u(t-3)+\sin t. \end{aligned} \tag{317}$$

The inverse transform of $\frac{1}{s\,(s^2+1)}$ can be computed via method of partial fractions, or better, via Convolution Theorem:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}*\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=1*\sin t=1-\cos t. \tag{318}$$

Thus

$$y=[1-\cos(t-3)]\,u(t-3)+\sin t. \tag{319}$$

Example 58. Solve

$$y''+5\,y'+6\,y=g(t), \qquad y(0)=0, \quad y'(0)=2, \tag{320}$$

where

$$g(t)=\begin{cases} 0 & 0\leqslant t<1 \\ t & 1<t<5 \\ 1 & 5<t \end{cases}. \tag{321}$$

Solution.

1. Transform the equation.

   a. Transform the LHS. Compute

$$\mathcal{L}\{y''\}=s^2\,Y-s\,y(0)-y'(0)=s^2\,Y-2; \tag{322}$$

$$\mathcal{L}\{y'\}=s\,Y-y(0)=s\,Y. \tag{323}$$

   So

$$\mathcal{L}\{y''+5\,y'+6\,y\}=\left(s^2+5\,s+6\right)Y-2. \tag{324}$$

   b. Transform the RHS. We first need to represent g using unit step functions. There are two discontinuities at $t=1,5$. We write

$$g(t)=0+A(t)\,u(t-1)+B(t)\,u(t-5) \tag{325}$$

   and determine

$$A(t)=t, \qquad B(t)=1-t. \tag{326}$$

   So

$$g(t)=t\,u(t-1)+(1-t)\,u(t-5). \tag{327}$$

   Recalling

$$\mathcal{L}\{g(t)\,u(t-a)\}=e^{-as}\,\mathcal{L}\{g(t+a)\}(s) \tag{328}$$

   We compute

$$\begin{aligned} \mathcal{L}\{t\,u(t-1)+(1-t)\,u(t-5)\} &= e^{-s}\,\mathcal{L}\{t+1\}+e^{-5s}\,\mathcal{L}\{-t-4\} \\ &= e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)-e^{-5s}\left(\frac{1}{s^2}+\frac{4}{s}\right). \end{aligned} \tag{329}$$

Thus the transformed equation reads

$$\left(s^2+5\,s+6\right)Y=e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)-e^{-5s}\left(\frac{1}{s^2}+\frac{4}{s}\right)+2. \tag{330}$$

2. Solve the transformed equation. We have

$$Y(s)=e^{-s}\,\frac{s+1}{s^2\,(s^2+5\,s+6)}-e^{-5s}\,\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}+\frac{2}{s^2+5\,s+6}. \tag{331}$$

3. Transform back. We need to use the formula

$$\mathcal{L}^{-1}\left\{e^{-as}\,F(s)\right\}=f(t-a)\,u(t-a). \tag{332}$$

Thus we have to compute the inverse transforms of $\frac{s+1}{s^2\,(s^2+5\,s+6)}$ and $\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}$. Due to linearity of the inverse transform, we first compute

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2+5\,s+6)}\right\} \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2+5\,s+6)}\right\}. \tag{333}$$

We have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{s^2\,(s^2+5\,s+6)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2+5\,s+6)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{(s+2)\,(s+3)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}*\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}\\
&= 1*\left(e^{-2t}-e^{-3t}\right\}\\
&= \int_0^t\left(e^{-2v}-e^{-3v}\right)\mathrm{d}v\\
&= \frac{1}{2}-\frac{1}{2}\,e^{-2t}-\frac{1}{3}+\frac{1}{3}\,e^{-3t}\\
&= \frac{1}{6}-\frac{1}{2}\,e^{-2t}+\frac{1}{3}\,e^{-3t}.
\end{aligned} \tag{334}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2+5\,s+6)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}*\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}\\
&= t*\left(e^{-2t}-e^{-3t}\right)\\
&= \int_0^t (t-v)\left(e^{-2v}-e^{-3v}\right)\mathrm{d}v\\
&= \int_0^t (t-v)\,\mathrm{d}\left(-\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\right)\\
&= (t-v)\left(-\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\right)\Big|_{v=0}^{v=t}-\int_0^t -\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\,\mathrm{d}(t-v)\\
&= \frac{1}{2}\,t-\frac{1}{3}\,t+\int_0^t\left(-\frac{1}{2}\,e^{-2v}+\frac{1}{3}\,e^{-3v}\right)\mathrm{d}v\\
&= \frac{1}{6}\,t+\frac{1}{4}\,e^{-2t}-\frac{1}{4}-\frac{1}{9}\,e^{-3t}+\frac{1}{9}\\
&= \frac{1}{6}\,t+\frac{1}{4}\,e^{-2t}-\frac{1}{9}\,e^{-3t}-\frac{5}{36}.
\end{aligned} \tag{335}$$

Thus we have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{1}{36}+\frac{t}{6}-\frac{1}{4}\,e^{-2t}+\frac{2}{9}\,e^{-3t} \tag{336}$$

which leads to

$$\mathcal{L}^{-1}\left\{e^{-s}\,\frac{s+1}{s^2\,(s^2+5\,s+6)}\right\}=\left(\frac{1}{36}+\frac{t-1}{6}-\frac{1}{4}\,e^{-2(t-1)}+\frac{2}{9}\,e^{-3(t-1)}\right)u(t-1); \tag{337}$$

For the other term we have

$$\mathcal{L}^{-1}\left\{\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}\right\}=\frac{19}{36}+\frac{t}{6}-\frac{7}{4}\,e^{-2t}+\frac{11}{9}\,e^{-3t} \tag{338}$$

which leads to

$$\mathcal{L}^{-1}\left\{e^{-5s}\,\frac{4\,s+1}{s^2\,(s^2+5\,s+6)}\right\}=\left(\frac{19}{36}+\frac{t-5}{6}-\frac{7}{4}\,e^{-2(t-5)}+\frac{11}{9}\,e^{-3(t-5)}\right)u(t-5). \tag{339}$$

Finally we quickly compute

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+5\,s+6}\right\}=2\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}=2\,e^{-2t}-2\,e^{-3t}. \tag{340}$$

Summarizing, we reach

$$\begin{aligned} y(t) \;&=\; \left(\frac{1}{36}+\frac{t-1}{6}-\frac{1}{4}\,e^{-2(t-1)}+\frac{2}{9}\,e^{-3(t-1)}\right)u(t-1) \\ &\quad -\left(\frac{19}{36}+\frac{t-5}{6}-\frac{7}{4}\,e^{-2(t-5)}+\frac{11}{9}\,e^{-3(t-5)}\right)u(t-5)+2\,e^{-2t}-2\,e^{-3t}. \end{aligned} \tag{341}$$

5.2. The Dirac delta function.

Definition 59. (Dirac Delta function) *The* ***Dirac delta function*** $\delta(t)$ *is characterized by the following two properties:*

$$\delta(t)=\begin{cases} 0 & t\neq 0 \\ \text{“infinity”} & t=0 \end{cases} \tag{342}$$

and

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,\mathrm{d}t=f(0) \tag{343}$$

for any function $f(t)$ that is continuous on an open interval containing $t=0$.

From the definition we conclude that

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,\mathrm{d}t=f(a) \tag{344}$$

for f that is continuous around a.

In fact, it is easy to be convinced that

$$\int_b^c f(t)\,\delta(t-a)\,\mathrm{d}t=\begin{cases} f(a) & b<a<c \\ 0 & a<b \text{ or } a>c \end{cases}. \tag{345}$$

Example 60. Evaluate

$$\int_{-\infty}^{\infty}\left(t^2-1\right)\delta(t)\,\mathrm{d}t. \tag{346}$$

Solution. We have

$$\int_{-\infty}^{\infty}\left(t^2-1\right)\delta(t)\,\mathrm{d}t=-1. \tag{347}$$

Example 61. Evaluate

$$\int_{-\infty}^{\infty} e^{-2t}\,\delta(t+1)\,\mathrm{d}t. \tag{348}$$

Solution. We have

$$\int_{-\infty}^{\infty} e^{-2t}\,\delta(t+1)\,\mathrm{d}t=e^{-2t}\,|_{t=-1}=e^2. \tag{349}$$

We can compute the Laplace transform of $\delta(t-a)$ when $a\geqslant 0$.

$$\mathcal{L}\{\delta(t-a)\}=\int_0^{\infty} e^{-st}\,\delta(t-a)\,\mathrm{d}t=e^{-as}. \tag{350}$$

In particular,

$$\mathcal{L}\{\delta(t)\} = 1. \tag{351}$$

Remark 62. Recalling Convolution Theorem

$$\mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{F\} * \mathcal{L}^{-1}\{G\}, \tag{352}$$

we conclude that

$$\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f\}\} = \delta(t-a) * f(t). \tag{353}$$

On the other hand, we already have the following formula using the unit step function:

$$\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f\}\} = f(t-a)\,u(t-a) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}. \tag{354}$$

For the whole Laplace transform theory to make sense, the two formulas must reconcile, that is we must show

$$\delta(t-a) * f(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}. \tag{355}$$

This can be shown as follows. We have

$$\delta(t-a) * f(t) = \int_0^t \delta(t-v-a)\, f(v)\, \mathrm{d}v. \tag{356}$$

Now we do a change of variable $s = t - v - a$, then $v = (t-a) - s$ and $\mathrm{d}v = -\,\mathrm{d}s$. Consequently

$$\int_0^t \delta(t-v-a)\, f(v)\, \mathrm{d}v = -\int_{t-a}^{-a} \delta(s)\, f((t-a)-s)\, \mathrm{d}s = \int_{-a}^{t-a} \delta(s)\, f((t-a)-s)\, \mathrm{d}s. \tag{357}$$

Now when $t > a$, $s = 0$ is contained in $(-a, t-a)$ while when $t < a$ $s = 0$ is not. Therefore we have

$$\int_{-a}^{t-a} \delta(s)\, f((t-a)-s)\, \mathrm{d}s = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases} = f(t-a)\, u(t-a). \tag{358}$$

Example 63. Determine

$$\mathcal{L}\{t\,\delta(t-1)\}. \tag{359}$$

Solution. We use definition:

$$\mathcal{L}\{t\,\delta(t-1)\} = \int_0^\infty e^{-st}\, t\,\delta(t-1)\, \mathrm{d}t = e^{-s}. \tag{360}$$

Furthermore we have

$$\int_{-\infty}^t \delta(x-a) = u(t-a). \tag{361}$$

Now consider a linear system

$$a\,y'' + b\,y' + c\,y = g(t), \qquad y(0) = y'(0) = 0. \tag{362}$$

We know that the transfer function is defined by

$$H(s) = \frac{Y(s)}{G(s)}. \tag{363}$$

In other words, we have

$$\mathcal{L}\{a\,h'' + b\,h' + c\,h\} = \left(a\,s^2 + b\,s + c\right) H(s) = 1. \tag{364}$$

This means the impulse response function $h(t)$ in fact solves

$$a\,h'' + b\,h' + c\,h = \delta(t). \tag{365}$$

Example 64. Solve

$$y'' + 2\,y' - 3\,y = \delta(t-1) - \delta(t-2). \qquad y(0) = 2, \quad y'(0) = -2. \tag{366}$$

Solution. We follow the three steps.

1. Transform the equation. We have

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2\,Y - s\,y(0) - y'(0) = s^2\,Y - 2\,s + 2; & (367)\\ \mathcal{L}\{y'\} &= s\,Y - y(0) = s\,Y - 2. & (368)\end{aligned}$$

Thus

$$\mathcal{L}\{y'' + 2\,y' - 3\,y\} = \left(s^2 + 2\,s - 3\right) Y - 2\,s - 2. \tag{369}$$

On the other hand

$$\mathcal{L}\{\delta(t-1) - \delta(t-2)\} = e^{-s} - e^{-2s}. \tag{370}$$

Thus the transformed equation reads

$$\left(s^2 + 2\,s - 3\right) Y - 2\,s - 2 = e^{-s} - e^{-2s}. \tag{371}$$

2. Solve the transformed equation. We have

$$Y = e^{-s}\,\frac{1}{s^2 + 2\,s - 3} - e^{-2s}\,\frac{1}{s^2 + 2\,s - 3} + 2\,\frac{s+1}{s^2 + 2\,s - 3}. \tag{372}$$

3. Transform back. We need to compute the inverse transform of $\frac{1}{s^2 + 2\,s - 3}$. We have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2\,s - 3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)\,(s-1)}\right\} = \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+3}\right\} = \frac{1}{4}\left(e^t - e^{-3t}\right). \tag{373}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\left\{e^{-s}\,\frac{1}{s^2 + 2\,s - 3}\right\} &= \frac{1}{4}\left(e^{t-1} - e^{3-3t}\right) u(t-1); & (374)\\ \mathcal{L}^{-1}\left\{e^{-2s}\,\frac{1}{s^2 + 2\,s - 3}\right\} &= \frac{1}{4}\left(e^{t-2} - e^{6-3t}\right) u(t-2). & (375)\end{aligned}$$

Finally we write

$$\frac{2\,s+2}{s^2 + 2\,s - 3} = \frac{A}{s-1} + \frac{B}{s+3} = \frac{(A+B)\,s + (3\,A - B)}{s^2 + 2\,s - 3} \Longrightarrow A = 1, B = 1. \tag{376}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{2\,s+2}{s^2 + 2\,s - 3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s+3}\right\} = e^t + e^{-3t}. \tag{377}$$

Putting everything together, we have

$$y(t) = \frac{1}{4}\left(e^{t-1} - e^{3-3t}\right) u(t-1) - \frac{1}{4}\left(e^{t-2} - e^{6-3t}\right) u(t-2) + e^t + e^{-3t}. \tag{378}$$

5.3. Gamma function.

Recall that for every integer $n \geqslant 0$, we have

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}. \tag{379}$$

Now what happens when n is not an integer? We compute (we use r instead of n to emphasize the possibility of r not being an integer)

$$\mathcal{L}\{t^r\}(s) \;=\; \int_0^\infty t^r\,e^{-st}\,\mathrm{d}t. \tag{380}$$

Setting

$$u = s\,t, \tag{381}$$

we have

$$\mathcal{L}\{t^r\}(s) = \int_0^\infty \left(\frac{u}{s}\right)^r e^{-u}\,\mathrm{d}\left(\frac{u}{s}\right) = \frac{1}{s^{r+1}} \int_0^\infty u^r\, e^{-u}\,\mathrm{d}u. \tag{382}$$

It turns out that in genearl the integral does not have a closed form representation. As such integrals are often seen in practice, we give it a name.

Definition 65. (Gamma function) *The **gamma function** $\Gamma(t)$ is defined by*

$$\Gamma(t) := \int_0^\infty e^{-u}\, u^{t-1}\,\mathrm{d}u, \qquad t > 0. \tag{383}$$

Thus

$$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}}. \tag{384}$$

Remark 66. One easily sees that the integral is well-defined for $t > 0$, but becomes infinity when $t \leqslant 0$. Nevertheless, the domain of the gamme function can be extended to $t \leqslant 0$ via the following recursive relation. Such things are discussed in texts of complex analysis.

Proposition 67. (Recursive relation for gamme functions) *We have*

$$\Gamma(t+1) = t\,\Gamma(t). \tag{385}$$

Proof. By definition

$$\begin{aligned} \Gamma(t+1) &= \int_0^\infty e^{-u}\, u^t\,\mathrm{d}u \\ &= -\int_0^\infty u^t\,\mathrm{d}e^{-u} \\ &= -u^t\, e^{-u}\,|_0^\infty + \int_0^\infty e^{-u}\,\mathrm{d}u^t \\ &= t\int_0^\infty e^{-u}\, u^{t-1}\,\mathrm{d}u \\ &= t\,\Gamma(t). \end{aligned} \tag{386}$$

Thus ends the proof. □

Example 68. Use the recursive relation and the fact that $\Gamma(1/2) = \sqrt{\pi}$, determine

a) $\mathcal{L}\left\{t^{-1/2}\right\}$,

b) $\mathcal{L}\left\{t^{7/2}\right\}$.

Solution. Recall that

$$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}}. \tag{387}$$

Thus

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}, \tag{388}$$

and

$$\mathcal{L}\left\{t^{7/2}\right\} = \frac{\Gamma(9/2)}{s^{9/2}} = \frac{\frac{7}{2}\Gamma(7/2)}{s^{9/2}} = \cdots = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma(1/2)}{s^{9/2}} = \frac{105}{16}\,\frac{\sqrt{\pi}}{s^{9/2}}. \tag{389}$$

6. Laplace transform of periodic functions.

Often we face problems involving periodic functions.

Definition 69. (Periodic function) *A function $f(t)$ is said to be* ***periodic of period T*** *if*

$$f(t+T)=f(t) \tag{390}$$

for all t in the domain of f.

For such function is determined by the values of f in any one period $[a, a+T)$. In particular, all information of f is contained in the values of $f(t)$ for $0 \leqslant t < T$. Naturally, we would like to compute its Laplace transform using these values only. Introduce the "windowed" version of $f(t)$:

$$f_T(t) := \begin{cases} f(t) & 0<t<T \\ 0 & \text{otherwise} \end{cases}. \tag{391}$$

Then we can compute the Laplace transform of f_T:

$$F_T(s) := \int_0^\infty e^{-st} f_T(t)\,\mathrm{d}t = \int_0^T e^{-st} f(t)\,\mathrm{d}t. \tag{392}$$

Now we would like to know the relation between F_T and $F=\mathcal{L}\{f\}$.

Theorem 70. *If f has period T and is piecewise continuous on $[0,T]$, then*

$$\mathcal{L}\{f\}(s)=\frac{F_T(s)}{1-e^{-sT}}. \tag{393}$$

Proof. We compute

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\ &= \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_T^{2T} e^{-st} f(t)\,\mathrm{d}t + \cdots \\ &= \sum_{n=0}^\infty I_n \end{aligned} \tag{394}$$

where

$$I_n = \int_{nT}^{(n+1)T} e^{-st} f(t)\,\mathrm{d}t. \tag{395}$$

Note that $I_0 = F_T$.

To evaluate I_n, we introduce a new variable $t_n := t - nT$. Then

$$I_n = \int_0^T e^{-s(t_n+nT)} f(t_n+nT)\,\mathrm{d}t_n = e^{-snT}\int_0^T e^{-st_n} f(t_n)\,\mathrm{d}t_n = e^{-snT} I_n = e^{-snT} F_T. \tag{396}$$

Therefore

$$\mathcal{L}\{f\}(s) = \sum_{n=0}^\infty I_n = F_T \left(\sum_{n=0}^\infty e^{-snT}\right) = \frac{F_T}{1-e^{-sT}}. \tag{397}$$

Thus ends the proof. □

Example 71. Determine $\mathcal{L}\{f\}$ where f has period 2 and is given by

$$f(t)=t, \qquad \text{on } 0<t<2. \tag{398}$$

Solution. We have

$$
\begin{aligned}
\mathcal{L}\{f\}(s) &= \frac{F_T(s)}{1-e^{-sT}} \\
&= \frac{\int_0^2 t\,e^{-st}\,\mathrm{d}t}{1-e^{-2s}} \\
&= \frac{1}{1-e^{-2s}}\left(-\frac{1}{s}\int_0^2 t\,\mathrm{d}e^{-st}\right) \\
&= -\frac{1}{s\,(1-e^{-2s})}\left[t\,e^{-st}\,|_{t=0}^{t=2} - \int_0^2 e^{-st}\,\mathrm{d}t\right] \\
&= -\frac{1}{s\,(1-e^{-2s})}\left[2\,e^{-2s} + \frac{1}{s}\,e^{-st}\,|_{t=0}^{t=2}\right] \\
&= -\frac{1}{s\,(1-e^{-2s})}\left[2\,e^{-2s} + \frac{1}{s}\,e^{-2s} - \frac{1}{s}\right] \\
&= \frac{1-e^{-2s}-2\,s\,e^{-2s}}{s^2\,(1-e^{-2s})}.
\end{aligned}
\tag{399}
$$