Solving Second Order Linear ODEs

1. Introduction: A few facts

We make a few things clear in this short introduction.

1. We will discuss linear equations only, as nonlinear equations of order two and higher are too hard.¹

2. Contrary to first order equations, there is no clever transformation that leads to direct integration. For example, there is no way to transform the equation

\[ y'' - 3y' - 4y = f(x) \]  

into the directly integrable form

\[ Y'' = F(X) \]

through introducing new unknown \( Y \) and new variable \( X \).

3. Linear homogeneous equations with constant coefficients are easy. Such equations of order 2 are very very easy. Such equations of order higher than 2 are reasonably easy. In one word, easy.

4. Linear nonhomogeneous equations with constant coefficients are conceptually still easy, but the calculation becomes complicated.

5. Linear equations with variable coefficients are hard. We will discuss them in the “Series method” section.

6. In theory, there is not much difference between 2nd order and higher order equations.

We will start from 2nd order constant coefficient equations.

2. Linear, constant-coefficient, homogeneous: exp is all we need.

That is, general solution to

\[ ay'' + by' + cy = 0 \]

where \( a, b, c \) are constants.

¹ A special class of 2nd order equation \( y'' = f(y, y') \) – those that \( x \) does not appear explicitly – can be transformed to 1st order by setting \( v = y' \). Then we have

\( \frac{dy}{dx} = v; \quad \frac{dv}{dx} = f(y, v) \)

which leads to

\( \frac{dy}{dv} = \frac{v}{f(y, v)} \).
Side: Exploring linearity.

The key property that enables us to solve a linear homogeneous equation is the following.

**Proposition 1.** If \( y_1, y_2, \ldots, y_k \) solves a linear homogeneous equation, so does \( c_1 y_1 + c_2 y_2 + \cdots + c_k y_k \) for any constants \( c_1, \ldots, c_k \).

Thus there is possibility of representing a large amount (actually infinitely many – as can be easily seen) of solutions by only a few. Now we ask

1. Is it possible to represent all solution by the linear combinations of finitely many?
2. If yes, how many is needed?

The answers are: 1. Yes; 2. Same as the order of the equation.

Therefore, to find the general solution for

\[
a_2(x) y'' + a_1(x) y' + a_0(x) y = 0
\]

we only need to find two solutions \( y_1, y_2 \), and then the general solution is

\[
c_1 y_1 + c_2 y_2.
\]

Or in the general case

\[
a_n(x) y^{(n)} + \cdots + a_0(x) y = 0
\]

we only need to find \( n \) solutions \( y_1, \ldots, y_n \) and then write

\[
c_1 y_1 + \cdots + c_n y_n.
\]

However, there is a catch.

**The \( n \) solutions must be linearly independent.**

**Definition 2.** (Linear dependence/independence) \( k \) functions \( y_1, \ldots, y_k \) are said to be linearly dependent if there are constants \( c_1, \ldots, c_k \) such that

\[
c_1 y_1(x) + \cdots + c_k y_k(x) = 0.
\]

Otherwise it is said to be linearly independent.

Checking linear independence is in general no easy task. But our case is special: \( y_1, \ldots, y_k \) are not just arbitrary functions, they are solutions to linear ODEs. As a consequence, there is a neat way of checking linear dependence/independence through some magical quantity called the “Wronskian”. We will discuss the Wronskian in a few days.

Now back to solving

\[
a y'' + b y' + c y = 0
\]

where \( a, b, c \) are constants. Recall that all we need to do is to find two solutions that are linearly independent. Note that it doesn’t matter how we obtain them.

It turns out we will obtain the two solutions through considering solutions of the special form \( y = e^{rx} \) for some constant \( r \). Substituting this into the equation leads to

\[
e^{rt} \left[ a r^2 + b r + c \right] = 0.
\]

Therefore, \( y = e^{rt} \) solves the equation if and only if \( a r^2 + b r + c = 0 \), or \( r \) is either

\[
r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{or} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\]
Recall from the theory of quadratic equations that there are three cases: Both \( r_1, r_2 \) real and \( r_1 \neq r_2; r_1 = r_2 \) real; \( r_1, r_2 \) complex. We discuss them one by one.

2.1. \( r_1 \neq r_2 \) real.

This is the easiest case. We have two solutions \( e^{r_1 t} \) and \( e^{r_2 t} \). If they are linearly independent, then all solutions to

\[
a y'' + b y' + c y = 0
\]

(14)
can be written as

\[
y = c_1 e^{r_1 t} + c_2 e^{r_2 t}
\]

(15)

and the equation is solved.

But are they linearly independent? Yes they are. We can either use the to-be-introduced Wronskian or prove directly.\(^2\)

**Example 3.** Solve

\[
y'' + 5 y' + 6 y = 0.
\]

(21)

**Solution.** Substituting \( y = e^{rt} \) into the equation, we reach

\[
r^2 + 5 r + 6 = 0 \implies r_1 = -3, r_2 = -2.
\]

(22)

As a consequence, the general solution is given by

\[
y = c_1 e^{-3t} + c_2 e^{-2t}.
\]

(23)

2.2. \( r_1 = r_2 \) real.

In this case \( e^{r_1 t} = e^{r_2 t} \), that is we can only find one solution of the form \( e^{rt} \). We need another solution which is linearly independent. It turns out that this other solution can be taken to be \( t e^{rt} \).\(^3\) It is easy to show that \( e^{rt} \) and \( t e^{rt} \) are linearly independent. Thus when \( r_1 = r_2 = r \) real, the general solution to

\[
a y'' + b y' + c y = 0
\]

(24)
is given by

\[
y = c_1 e^{rt} + c_2 t e^{rt}.
\]

(25)

**Example 4.** Solve

\[
y'' + 8 y' + 16 y = 0.
\]

(26)

**Solution.** Substituting \( y = e^{rt} \) we easily see that

\[
r^2 + 8 r + 16 = 0 \implies r_1 = r_2 = -4.
\]

(27)

---

2. Assume that there are constants \( C_1, C_2 \) such that

\[
0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}.
\]

(16)

Differentiating with respect to \( t \), we have

\[
0 = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}.
\]

(17)

In other words, the constants \( C_1, C_2 \) solve the system

\[
e^{r_1 t} x + e^{r_2 t} y = 0
\]

(18)

\[
r_1 e^{r_1 t} x + r_2 e^{r_2 t} y = 0.
\]

(19)

Solving this system shows that the only solutions are \( x = y = 0 \). Therefore \( C_1 = C_2 = 0 \). We have shown that

If \( C_1, C_2 \) are such that \( 0 = C_1 e^{r_1 t} + C_2 e^{r_2 t} \), then \( C_1 = C_2 = 0 \).

(20)

This is exactly the linearly independence of \( e^{r_1 t} \) and \( e^{r_2 t} \).

3. Note that \( t e^{rt} = \frac{d}{dt}(e^{rt}) \). This is not a coincidence! Hint: In the case of \( r_1 \neq r_2 \), \( e^{r_1 t} \), \( e^{r_2 t} \) are solutions, so is \( e^{r_1 t} - e^{r_2 t} \).

(21)

4. Another way is to use “reduction of order”. That is we look for the 2nd solution of the form \( u(t) e^{rt} \). Then the equation for \( u \) will be just first order and can be easily solved, yielding \( u(t) = a t + b \).
Thus the general solution is given by
\[ y = c_1 e^{-4t} + c_2 t e^{-4t}. \] (28)

2.3. \( r_1, r_2 \) complex.

As
\[ r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \] (29)
with \( a, b, c \) real, the only possibility for \( r_1, r_2 \) to be complex is that
\[ r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta \] (30)
where \( \alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac - b^2}}{2a} \) are both real.

In this case, what we have are two linearly independent complex solutions
\[ e^{(\alpha+i\beta)t}, \quad e^{(\alpha-i\beta)t}. \] (31)
The task now is to produce two linearly independent real solutions out of them.

The idea is to use the properties of complex exponentials together with Euler’s formula:
\[ e^{i\theta} = \cos \theta + i \sin \theta \] (32)
to obtain
\[ e^{(\alpha+i\beta)t} = e^{\alpha t} \cos \beta t + i e^{\alpha t} \sin \beta t, \quad e^{(\alpha-i\beta)t} = e^{\alpha t} \cos \beta t - i e^{\alpha t} \sin \beta t. \] (33)
Recall that due to the linearity of the equation, if \( y_1, y_2 \) are solutions, so is \( c_1 y_1 + c_2 y_2 \) for any constants \( c_1, c_2 \). Inspecting the formulas for \( e^{(\alpha\pm i\beta)t} \) we easily figure out that
\[ e^{\alpha t} \cos \beta t = \frac{1}{2} e^{(\alpha+i\beta)t} + \frac{1}{2} e^{(\alpha-i\beta)t} \] (34)
and
\[ e^{\alpha t} \sin \beta t = \frac{1}{2} e^{(\alpha+i\beta)t} - \frac{1}{2} e^{(\alpha-i\beta)t} \] (35)
are also solutions.

Now we show that they are linearly independent. Assume that \( C_1, C_2 \) are constants such that
\[ C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t = 0. \] (36)
This gives
\[ \frac{C_1 + C_2}{2} e^{(\alpha+i\beta)t} + \frac{C_1 - C_2}{2} e^{(\alpha-i\beta)t} = 0. \] (37)
As \( e^{(\alpha\pm i\beta)t} \) are linearly independent, necessarily
\[ \frac{C_1 + C_2}{2} = \frac{C_1 - C_2}{2} = 0 \] (38)
from which \( C_1 = C_2 = 0 \) immediately follows.

Example 5. Solve
\[ y'' + 9y = 0. \] (39)

Solution. Substituting \( y = e^{rt} \) into the equation, we reach
\[ r^2 + 9 = 0 \implies r_1 = 3i, \quad r_2 = -3i. \] (40)
Thus we obtain two complex solutions
\[ e^{3it} = \cos 3t + i \sin 3t \quad \text{and} \quad e^{-3it} = \cos 3t - i \sin 3t. \] (41)
From this we obtain two real solutions
\[ \cos 3t, \sin 3t. \] (42)
The general solution is then given by

\[ y = c_1 \cos 3t + c_2 \sin 3t. \]

(43)

Side: Checking linear independence, the Wronskian.

Given two solutions \( y_1, y_2 \), we can compute their Wronskian, defined as the determinant:

\[
W(y_1, y_2) := \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix}.
\]

(44)

Then

- \( y_1, y_2 \) are linearly dependent over an interval \( I \) if and only if \( W(y_1, y_2) \equiv 0 \) over \( I \).\(^5\)

When \( y_1, y_2 \) solves a 2nd order linear homogeneous ODE

\[ y'' + p(t) y' + q(t) y = 0, \]

(45)

we have furthermore Abel’s Theorem (assuming \( p(t) \) is continuous over the interval \( I \) under consideration):

\[
W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp \left[ -\int_{t_0}^{t} p(s) \, ds \right]
\]

(46)

which enables us to find out the value of the Wronskian at any point without actually solving the equation. Furthermore, Abel’s Theorem implies

- Either \( W(y_1, y_2) \equiv 0 \) for all \( x \in I \), or \( W(y_1, y_2) \not\equiv 0 \) for any \( x \in I \).

Finally, the above remains true for higher order equations, where we define the Wronskian as

\[
W(y_1, \ldots, y_n) := \det \begin{pmatrix} y_1 & \cdots & y_n \\ y'_1 & \cdots & y'_n \\ \vdots & \ddots & \vdots \\ y^{(n-1)}_1 & \cdots & y^{(n-1)}_n \end{pmatrix}
\]

(47)

where \( n \) is the order of the equation.

2.4. More examples.

We first recall the procedure of solving linear 2nd order constant-coefficient equations:

\[ a y'' + b y' + c y = 0. \]

(48)

1. Write down and solve the auxiliary equation

\[ a r^2 + b r + c = 0 \iff r_1, r_2. \]

(49)

Or equivalently, substitute \( y = e^{rt} \) into the equation.

2. Three cases.

   a. \( r_1 \neq r_2 \), both real. General solution is given by

\[ y = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \]

(50)

   b. \( r_1 = r_2 \) real. General solution is given by

\[ y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}. \]

(51)

5. This is not true if \( y_1, y_2 \) are not solutions to the same linear ODE. That is, if \( y_1, y_2 \) are just two functions, \( W(y_1, y_2) \equiv 0 \not\implies y_1, y_2 \) linearly dependent!
c. \( r_1, r_2 \) complex. In this case \( r_1 = \alpha + i \beta, r_2 = \alpha - i \beta \) and the general solution is given by

\[
y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.
\]  

(52)

Now we look at more examples.

**Example 6.** Solve

\[
y'' - y' - 11 y = 0.
\]  

(53)

**Solution.** Try \( y = e^{rt} \). We have

\[
r^2 - r - 11 = 0 \implies r_1 = \frac{1 + \sqrt{1 + 44}}{2} = \frac{1 + 3\sqrt{5}}{2}, \quad r_2 = \frac{1 - 3\sqrt{5}}{2}.
\]  

(54)

As a consequence, the general solution is given by

\[
y = c_1 e^{\frac{1 + 3\sqrt{5}}{2} t} + c_2 e^{\frac{1 - 3\sqrt{5}}{2} t}.
\]  

(55)

**Example 7.** Solve

\[
y'' + 2 y' - 8 y = 0; \quad y(0) = 3, \quad y'(0) = -12
\]  

(56)

**Solution.** First we find the general solution. Substituting \( y = e^{rt} \) leads to

\[
r^2 + 2r - 8 = 0 \implies r_1 = -4, r_2 = 2.
\]  

(57)

So the general solution is given by

\[
y(t) = c_1 e^{-4t} + c_2 e^{2t}.
\]  

(58)

To determine the constants we use the initial values:

\[
3 = y(0) = c_1 + c_2; \quad -12 = y'(0) = -4c_1 + 2c_2.
\]  

(59)

Solving this linear system (with \( c_1, c_2 \) as unknowns) gives

\[
c_1 = 3, \quad c_2 = 0.
\]  

(60)

So finally the solution to the initial value problem is given by

\[
y(t) = 3 e^{-4t}.
\]  

(61)

**Example 8.** Solve

\[
4 w'' + 20 w' + 25 w = 0.
\]  

(62)

**Solution.** Substituting \( w = e^{rt} \) leads to

\[
4r^2 + 20r + 25 = 0 \implies r_1 = r_2 = -\frac{5}{2}
\]  

(63)

Thus the two linearly independent solutions are

\[
e^{-\frac{5}{2}t} \text{ and } te^{-\frac{5}{2}t}.
\]  

(64)

As a consequence, the general solution is

\[
w(t) = c_1 e^{-\frac{5}{2}t} + c_2 t e^{-\frac{5}{2}t}.
\]  

(65)

**Example 9.** Solve

\[
y'' + 2 y' + y = 0; \quad y(0) = 1, \quad y'(0) = -3.
\]  

(66)

**Solution.** First we find the general solution. Substituting \( y = e^{rt} \) leads to

\[
r^2 + 2r + 1 = 0 \implies r_1 = r_2 = -1.
\]  

(67)
Thus the general solution is given by
\[ y(t) = c_1 e^{-t} + c_2 t e^{-t}. \] (68)

Using the initial values:
\[ 1 = y(0) = c_1; \quad -3 = y'(0) = -c_1 + c_2. \] (69)

This gives
\[ c_1 = 1, \quad c_2 = -2. \] (70)

Therefore the solution to the initial value problem is given by
\[ y(t) = 1 e^{-t} - 2 t e^{-t}. \] (71)

**Example 10.** Solve
\[ y'' + 2 y' + 5 y = 0. \] (72)

**Solution.** Substituting \( y = e^{rt} \) leads to
\[ r^2 + 2r + 5 = 0 \Rightarrow r_1 = -1 + 2i, \quad r_2 = -1 - 2i. \] (73)

The two linearly independent complex solutions are then given by
\[ e^{(-1+2i)t} = e^{-t} [\cos 2t + i \sin 2t], \quad e^{(-1-2i)t} = e^{-t} [\cos 2t - i \sin 2t]. \] (74)

From this we can obtain the real solutions:
\[ e^{-t} \cos 2t, \quad e^{-t} \sin 2t. \] (75)

Finally the general solution is given by
\[ y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t. \] (76)

**Example 11.** Solve
\[ y'' + 2 y' + 2 y = 0; \quad y(0) = 2, \quad y'(0) = 1. \] (77)

**Solution.** First we find the general solution. Substituting \( y = e^{rt} \) gives
\[ r^2 + 2r + 2 = 0 \Rightarrow r_1 = -1 + i, \quad r_2 = -1 - i. \] (78)

The complex solutions are then
\[ e^{(-1+i)t}, e^{(-1-i)t} \] (79)

which yield the real solutions
\[ e^{-t} \cos t, \quad e^{-t} \sin t. \] (80)

So the general solution is given by
\[ y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t. \] (81)

Now we use the initial values:
\[ 2 = y(0) = c_1, \quad 1 = y'(0) = -c_1 + c_2. \] (82)

Solving this we obtain
\[ c_1 = 2, \quad c_2 = 3. \] (83)

The solution to the initial value problem is then given by
\[ y(t) = 2 e^{-t} \cos t + 3 e^{-t} \sin t. \] (84)

**2.5. Linear, homogeneous, constant coefficient equations of higher order.**

The above theory and method naturally extends to linear homogeneous equations of higher orders:
\[ a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \] (85)
We have

- If \( y_1, \ldots, y_n \) are solutions that are linearly independent\(^6\), then the general solution is given by
  \[
  y = c_1 y_1 + \cdots + c_n y_n.
  \]  
  \[
  \cdots (86)
  \]

- To find \( y_1, \ldots, y_n \), we try \( y = e^{rt} \) to reach
  \[
  a_n r^n + \cdots + a_1 r + a_0 = 0 \implies r = r_1, \ldots, r_n
  \]  
  \[
  \cdots (87)
  \]
  Then if \( r_1, \ldots, r_n \) are all different and real,
  \[
  e^{r_1 t}, \ldots, e^{r_n t}
  \]  
  \[
  \cdots (88)
  \]
  are the \( n \) solutions we want.

- If some \( r_i \)'s coincide, or some are complex, we use similar tricks. If \( r \) is a root of multiplicity \( m \), then we put
  \[
  e^{rt}, t e^{rt}, \ldots, t^{m-1} e^{rt}
  \]  
  \[
  \cdots (89)
  \]
  in our list of linearly independent solutions. If \( \alpha \pm i \beta \) are complex roots, then we put
  \[
  e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t
  \]  
  \[
  \cdots (90)
  \]
  into our list of linearly independent solutions.\(^7\)

- There is a new situation here: What if some \( r_i \)'s are both multiple roots and complex? The trick is still the same. For example, suppose \( \alpha + i \beta \) is a double root (note that \( \alpha - i \beta \) must also be a double root). Then the four solutions corresponding to \( \alpha \pm i \beta \) are:
  \[
  e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, t e^{\alpha t} \cos \beta t, t e^{\alpha t} \sin \beta t.
  \]  
  \[
  \cdots (91)
  \]
  This is a result of
  \[
  e^{(\alpha \pm i \beta) t}, t e^{(\alpha \pm i \beta) t}
  \]  
  \[
  \cdots (92)
  \]
  being solutions.

**Example 12.** Solve

\[
 y''' + y'' - 6 y' + 4 y = 0.
 \]  
  \[
  \cdots (93)
  \]

**Solution.** Substituting \( y = e^{rt} \) gives

\[
 r^3 + r^2 - 6 r + 4 = 0.
 \]  
  \[
  \cdots (94)
  \]

Inspection suggests \( r = 1 \) is a solution. Thus \( r^3 + r^2 - 6 r + 4 = (r - 1) \left[ r^2 + 2 r - 4 \right] = (r - 1) \left[ \left( r - \left( -1 + \sqrt{5} \right) \right) \left( r - \left( -1 - \sqrt{5} \right) \right) \right] \).

Thus we have three different roots:

\[
 r_1 = 1, \quad r_2 = -1 + \sqrt{5}, \quad r_3 = -1 - \sqrt{5}.
 \]  
  \[
  \cdots (95)
  \]

The general solution is then given by

\[
 y(t) = c_1 e^t + c_2 e^{(-1+\sqrt{5})t} + c_3 e^{(-1-\sqrt{5})t}.
 \]  
  \[
  \cdots (96)
  \]

**Example 13.** Solve

\[
 z''' + 2 z'' - 4 z' - 8 z = 0.
 \]  
  \[
  \cdots (97)
  \]

**Solution.** Substituting \( z = e^{rt} \) leads to

\[
 r^3 + 2 r^2 - 4 r - 8 = 0.
 \]  
  \[
  \cdots (98)
  \]

\[
 6. \text{ Meaning: If constants } C_1, \ldots, C_n \text{ makes } C_1 y_1 + \cdots + C_n y_n = 0, \text{ then } C_1 = C_2 = \cdots = C_n = 0.
\]

\[
 7. \text{ It can be shown that if } \alpha + i \beta \text{ is a root, so is } \alpha - i \beta. \text{ That is, complex roots have to appear in conjugate pairs (otherwise we are in trouble!).}
\]
Inspecting for a while we realize that
\[ r^3 + 2r^2 - 4r - 8 = (r + 2)(r^2 - 4) = (r + 2)^2 (r - 2). \]  
(100)
Thus the roots are
\[ r_1 = r_2 = -2, \quad r_3 = 2 \]  
(101)
and the general solution is given by
\[ z(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 e^{2t}. \]  
(102)

3. Nonhomogeneous equations – how to find the particular solution.

Having understood linear 2nd order constant-coefficient homogeneous equations, we turn to equations with nonzero right hand side:
\[ ay'' + by' + cy = f(t). \]  
(103)
Now the crucial observation is the following. Let \( y_p \) be a solution to this equation. Then for any other solution \( \tilde{y}, \ y := \tilde{y} - y_p \) solves
\[ ay'' + by' + cy = 0. \]  
(104)
In other words, once we find one solution to the nonhomogeneous equation (this solution will be called “particular solution”), what remains is just solving the homogeneous equation.

More specifically, if \( y_p(t) \) is a particular solution, \( y_1, y_2 \) are two linearly independent solutions to the homogeneous equation, then the general solution for the nonhomogeneous equation is given by
\[ y = c_1 y_1 + c_2 y_2 + y_p \]  
(105)
As we have already know how to solve the homogeneous equation, the question is

How to find the particular solution \( y_p \)?

It turns out that finding this one solution is quite difficult. There is a general method called “variation of parameters”. However the calculation involved may get a bit messy in practice – in particular, it involves integration of possibly complicated functions. On the other hand, for several class of special \( f(t) \)’s, one can take “short-cuts” by cleverly guess the form of the solution. In the following we will first discuss these special cases, and then turn to the general method.

3.1. The method of undetermined coefficients.

The basic idea is as follows. Since functions like \( e^{at}, \{\sin \beta t, \cos \beta t\}, \{t^n\} \) are “closed” under differentiation, when such functions appear on the right hand side, we guess that there are solutions of similar form.

Example 14. Find a particular solution of
\[ 2 z'' + z = 9 e^{2t}. \]  
(106)
Solution. We guess a solution of the form \( z = A e^{2t} \). Substituting into the equation we have
\[ 2 z'' + z = 8 A e^{2t} + A e^{2t} = 9 A e^{2t}. \]  
(107)
It is clear that taking \( A = 9/5 \) leads to a particular solution
\[ z(t) = e^{2t}. \]  
(108)

Example 15. Find a particular solution of
\[ 2 x' + x = 3 t^2. \]  
(109)
Solution. As \( (t^n)' = n t^{n-1} \), we would like to guess that there is a particular solution of the form \( A t^n \). Substituting into the left hand side, we find
\[ 2 x' + x = 2 A n t^{n-1} + A t^n. \]  
(110)
This clearly cannot be equal to the right hand side. But this first try tells us two things:

1. To obtain \(3t^2\), we cannot use a single power \(t^n\). We have to use a polynomial \(A_n\ t^n + A_{n-1} t^{n-1} + \ldots\).

2. The highest power on the left hand side is \(t^n\), coming from \(x\). It cannot be cancelled by any term coming from \(x'\). As a consequence, \(n = 2\).

Therefore we would like to try

\[
z = At^2 + Bt + C. \tag{111}\]

Substituting into the equation, we have

\[
2x' + x = 2[2At + B] + At^2 + Bt + C = At^2 + (4A + B)t + (2B + C). \tag{112}\]

Equating with the right hand side:

\[
At^2 + (4A + B)t + (2B + C) = 3t^2 \tag{113}\]

gives the following system:

\[
\begin{align*}
A &= 3 \tag{114} \\
4A + B &= 0 \tag{115} \\
2B + C &= 0 \tag{116}
\end{align*}
\]

whose solution is

\[A = 3, \quad B = -12, \quad C = 24. \tag{117}\]

As a consequence, the particular solution can be taken as

\[x_p(t) = 3t^2 - 12t + 24. \tag{118}\]

**Example 16.** Solve

\[y'' - y' + 9y = 3 \sin 3t. \tag{119}\]

**Solution.** Recall that differentiating \(\sin\) one obtains \(\cos\), and differentiating \(\cos\) one obtains \(\sin\). Thus the set \(\{A \sin 3t + B \cos 3t\}\) is closed under differentiations.

Inspired by this, we guess

\[y = A \sin 3t + B \cos 3t. \tag{120}\]

Substituting into the left hand side, we have

\[
y'' - y' + 9y = [ -9 A \sin 3t - 9 B \cos 3t ] - [ 3A \cos 3t - 3B \sin 3t ] + 9[ A \sin 3t + B \cos 3t] = 3B \sin 3t - 3A \cos 3t. \tag{121}\]

Equating with the right hand side:

\[3B \sin 3t - 3A \cos 3t = 3 \sin 3t \tag{122}\]

we obtain

\[B = 1, \quad A = 0. \tag{123}\]

Thus the particular solution is given by

\[y_p(t) = \cos 3t. \tag{124}\]

**Example 17.** Find a particular solution for

\[y''(x) + y(x) = 2^x. \tag{125}\]

**Solution.** Writing \(2^x = e^{(\ln 2)x}\), we naturally guess

\[y(x) = A e^{(\ln 2)x}. \tag{126}\]
Substituting into the equation, we reach
\[ y'' + y = A (\ln 2)^2 e^{(\ln 2)x} + A e^{(\ln 2)x} = A \left[ (\ln 2)^2 + 1 \right] 2^x. \] (127)
Equating with the right hand side, we have
\[ A = \left[ (\ln 2)^2 + 1 \right]^{-1} \] (128)
which leads to
\[ y_p(x) = \left[ (\ln 2)^2 + 1 \right]^{-1} e^{(\ln 2)x} = \left[ (\ln 2)^2 + 1 \right]^{-1} 2^x. \] (129)

Thus we can find the general solution to equations with right hand side of the above forms.

**Example.** Find the general solution to
\[ y'' - y = t. \] (130)

**Solution.** Recall that all we need to do is to find one particular solution and the general solution to the corresponding homogeneous equation
\[ y'' - y = 0. \] (131)

- Finding one particular solution. Try \( y = At + B \). Substituting into the left hand side, we have
\[ y'' - y = 0 - (At + B). \] (132)
Equating with the right hand side, we find out \( A = -1, B = 0 \). Thus \( y_p = -t \) is a particular solution.
- Finding general solution to the homogeneous equation. Substituting \( y = e^{rt} \) gives
\[ r^2 - 1 = 0 \implies r_1 = 1, r_2 = -1. \] (133)
Thus the general solution is given by
\[ \hat{y} = c_1 e^t + c_2 e^{-t}. \] (134)
Putting everything together, we see that the general solution to the non-homogeneous equation is
\[ y(t) = c_1 e^t + c_2 e^{-t} - t. \] (135)

**Example 18.** Find the general solution to
\[ 2 y'' + 3 y' - 4 y = 2 t + \sin 2 t + 3. \] (136)

**Solution.** The idea now is to use superposition principle, that is, if \( y_i \) solves
\[ a y'' + b y' + c y = f_i, \] (137)
then \( c_1 y_1 + \cdots + c_k y_k \) solves
\[ a y'' + b y' + c y = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k. \] (138)
Guided by this, what we need to do (to solve the problem) are the following: First we simplify the right hand side using the trigonometric formula
\[ \cos 2t = 1 - 2\sin^2 t \implies \sin^2 t = \frac{1}{2} - \frac{1}{2} \cos 2t. \] (139)
Thus the right hand side becomes
\[ 2 t - \frac{1}{2} \cos 2 t + \frac{7}{2}. \] (140)
We need to
- Find a particular solution \( y_{p1} \) for
\[ 2 y'' + 3 y' - 4 y = 2 t; \] (141)
• Find a particular solution $y_p_2$ for
\[ 2 y'' + 3 y' - 4 y = -\frac{1}{2} \cos 2t; \] (142)

• Find a particular solution $y_p_3$ for
\[ 2 y'' + 3 y' - 4 y = \frac{7}{2}; \] (143)

• Find the general solution $\tilde{y}$ for
\[ 2 y'' + 3 y' - 4 y = 0. \] (144)

Then the general solution to the nonhomogeneous equation is given by
\[ y = \tilde{y} + y_{p_1} + y_{p_2} + y_{p_3}. \] (145)

Now we carry this plan out.

• Find a particular solution $y_{p_1}$ for
\[ 2 y'' + 3 y' - 4 y = 2t; \] (146)

Based on the right hand side, we guess $y = At + B$. Substituting into the equation we reach
\[ -4At + 3A - 4B = 2t. \] (147)

Thus
\[ A = -\frac{1}{2}, \quad B = -\frac{3}{8} \] (148)

and
\[ y_{p_1} = -\frac{1}{2}t - \frac{3}{8}. \] (149)

• Find a particular solution $y_{p_2}$ for
\[ 2 y'' + 3 y' - 4 y = -\frac{1}{2} \cos 2t; \] (150)

We use the ansatz
\[ y = A \cos 2t + B \sin 2t. \] (151)

Then
\[
2 y'' + 3 y' - 4 y = 2[-4A \cos 2t - 4B \sin 2t] + 3[-2A \sin 2t + 2B \cos 2t]
- 4[A \cos 2t + B \sin 2t]
= (-12A + 6B) \cos 2t + (-12B - 6A) \sin 2t.
\] (152)

Equating with the right hand side
\[ -12A + 6B = -\frac{1}{2}; \quad -12B - 6A = 0. \] (153)

This gives
\[ A = \frac{1}{30}; \quad B = -\frac{1}{60}. \] (154)

Therefore we can take
\[ y_{p_2} = \frac{1}{30} \cos 2t - \frac{1}{60} \sin 2t. \] (155)

• Find a particular solution $y_{p_3}$ for
\[ 2 y'' + 3 y' - 4 y = \frac{7}{2}; \] (156)

As the right hand side is a constant, we try
\[ y = A. \] (157)

This quickly leads to
\[ y_{p_3} = -\frac{7}{8}. \] (158)

• Find the general solution $\tilde{y}$ for
\[ 2 y'' + 3 y' - 4 y = 0. \] (159)
Substituting \( y = e^{rt} \) leads to
\[
2r^2 + 3r - 4 = 0 \implies r_1 = -3 + \sqrt{31}, \quad r_2 = -3 - \sqrt{31}.
\] (160)

We have two distinct real roots, thus the general solution is given by
\[
y = c_1 e^{-3 + \sqrt{31} t} + c_2 e^{-3 - \sqrt{31} t}.
\] (161)

Putting everything together, the solution to the original problem is
\[
y = \tilde{y} + y_{p1} + y_{p2} + y_{p3} = c_1 e^{-3 + \sqrt{31} t} + c_2 e^{-3 - \sqrt{31} t} - \frac{1}{2} t + \frac{1}{30} \cos 2t - \frac{1}{60} \sin 2t - \frac{5}{4}.
\] (162)

From the above experience we see that, basically, when the right hand side is of the form \( e^{at} \), we guess the solution to be of the form \( A e^{at} \) (Note that constant right hand side belongs to this case): When the RHS is \( \sin \beta t \) or \( \cos \beta t \), we guess the solution to be \( A \cos \beta t + B \sin \beta t \); When the RHS is of the form \( t^m \), we guess the solution to be a polynomial of degree \( m \), that is \( A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0 \).

However life is not so simple, as is shown by the following example.

**Example 19.** (4.4.17) Solve
\[
y'' - 2y' + y = 8e^t.
\] (163)

**Solution.** As usual we try
\[
y = Ae^t.
\] (164)

Substituting into the equation, we have
\[
y'' - 2y' + y = A \left( (e^t)'' - 2e^t + e^t \right) = 0
\] (165)

which clearly cannot be made \( 8e^t \)! Inspecting the above, we see that the problem is that \( e^t \) is a solution to the corresponding homogeneous equation, thus everything cancelled out and we have nothing left to adjust.

The idea now is to put factor of the form \( t^k \) in front of \( e^t \). Let’s see what is the appropriate \( k \). Substituting \( y = At^k e^t \) we have
\[
y'' - 2y' + y = A \left[ t^k e^t + 2k t^{k-1} e^t + k(k-1) t^{k-2} e^t \right] - 2A \left( t^k e^t + k t^{k-1} e^t \right) + At^k e^t
\]
\[
= A t^k [e^t - 2e^t + e^t] + A [t^{k-1} \left( 2k e^t - 2ke^t \right) + A k (k-1) t^{k-2} e^t
\]
\[
= A k (k-1) t^{k-2} e^t.
\] (166)

As the right hand side is \( 8e^t \), we see that the appropriate \( k \) is 2.

Substituting \( y = At^2 e^t \) into the equation we reach
\[
2A e^t = 8e^t \implies A = 4.
\] (167)

So the desired particular solution is
\[
y = 4t^2 e^t.
\] (168)

Looking at the above example again, we see that \( e^t \) corresponds to \( r = 1 \), which is a double root of the auxiliary equation
\[
r^2 - 2r + 1 = 0
\] (169)

of the homogeneous equation
\[
y'' - 2y' + y = 0.
\] (170)

That is, the multiplicity of the root \( r = 1 \) is 2. We notice that this is exactly the value of \( k \) we need to take in the ansatz \( t^k e^t \). **This is no coincidence!**
Now we introduce the full version of the method.

**Method of undetermined coefficients**

To find a particular solution to the differential equation

\[ a y'' + b y' + c y = P_m(t) e^{rt}, \]  

(171)

where \( P_m(t) \) is a polynomial of degree \( m \), use the form

\[ y_p(t) = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{rt}; \]  

(172)

where

- \( r \) is not a root of the associated auxiliary equation \( \implies s = 0; \)
- \( r \) is a simple root \( \implies s = 1; \)
- \( r \) is a double root \( \implies s = 2. \)

To find a particular solution to the differential equation

\[ a y'' + b y' + c y = P_m(t) e^{\alpha t} \cos \beta t + Q_n(t) e^{\alpha t} \sin \beta t \]  

(173)

where \( P_m(t) \) is a polynomial of degree \( m \) and \( Q_n(t) \) is a polynomial of degree \( n \), use the form

\[ y_p(t) = t^s (A_k t^k + \cdots + A_1 t + A_0) e^{\alpha t} \cos \beta t \]
\[ + t^s (B_k t^k + \cdots + B_1 t + B_0) e^{\alpha t} \sin \beta t. \]  

(174)

where \( k \) is the larger of \( m \) and \( n \) (that is \( k = \max \{m, n\} \)), and \( s \) is determined by

- \( \alpha + \beta i \) is not a root of the associated auxiliary equation \( \implies s = 0; \)
- \( \alpha + \beta i \) is a root \( \implies s = 1.8 \)

**Side: Making sense of this mess.**

Somehow I feel the above is too mysterious, so I will try to make sense out of it. The following is what I would call “pseudo-math” in the sense of “pseudo-code” – looks like math argument but actually is not. Anyone who’s satisfied with the above rules can skip the following.

In the method of undetermined coefficients, we guess the form of the particular solution involving a number of unknown constant coefficients, plug it into the equation, and try to determine the value of these constants by equating the “same terms” on the left and the right. Each term yields an equation that has to be satisfied.

To be able to carry this out successfully, we need

The number of equations (for those unknown coefficients) is the same as the number of the unknown coefficients.

In particular, if the number of the unknown coefficients is less than the number of equations, we won’t be able to find our particular solution.

Now let’s count. Say let’s have

\[ y'' + 2 y = e^t. \]  

(175)

The right hand side is a single term \( e^t \). So we expect one equation. Therefore we guess \( y_p = A e^t \) and it works.

Of course things are not so simple. Consider

\[ y'' + 2 y = \sin t. \]  

(176)

8. Think: What happens to the “double root” case?
The right hand side is a single term \( \sin t \) and we guess \( y_p = A \sin t \) and it won’t work. Why? Because the real number of terms on the right hand side is not one, but two. Indeed, when counting the number of terms, we have to also count the “hidden ones” that can be obtained through differentiation.

In the above example, \( \sin t \) is the one that appears explicitly, but if we differentiate \( \sin t \), we obtain \( \cos t \), we have to count this one too. As differentiating \( \cos t \) we obtain \( \sin t \) again, the total number of terms on the right hand side is 2 and our \( y_p \) should be assumed as \( y_p = A \cos t + B \sin t \).

Same philosophy applies to polynomial right hand sides:

\[
y'' + y = t^{10}.
\] (177)

The right hand side is a polynomial, so we guess \( y_p \) is also a polynomial. But of what order? \( t^{10} \) is just one term, but differentiating it yields 10 “hidden terms” \( t^9, t^8, \ldots, t, 1 \). So there are 11 equations to be satisfied. We need 11 unknowns. So \( y_p = a_{10} t^{10} + \cdots + a_1 t + a_0, \) exactly 11 unknowns.

Now consider the general case

\[
a y'' + b y' + c y = P_m(t) e^{\alpha t} \cos \beta t + Q_n(t) e^{\gamma t} \sin \beta t.
\] (178)

We know

\[
y_p = (A_k t^k + \cdots + A_1 t + A_0) e^{\alpha t} \cos \beta t + (B_k t^k + \cdots + B_1 t + B_0) e^{\gamma t} \sin \beta t.
\] (179)

But what is \( k \)? We see that the right hand side has \( m + 1 + n + 1 \) explicit terms. But (assuming \( n > m \)) differentiating the second term yields \( n - m \) more “hidden” ones. So we expect a total of 2 \((n + 1)\) equations. So we should take \( k \) to be \( n = \max \{m, n\} \).

Finally the real test. Consider

\[
a y'' + b y' + c y = P_m(t) e^{\alpha t}
\] (180)

with \( r \) a single root. We see that the right hand side contains \( m + 1 \) terms. So we need \( m + 1 \) unknowns. Guess

\[
y_p = (A_m t^m + \cdots + A_1 t + A_0) e^{\alpha t}.
\] (181)

But it won’t work. Why? Because \( A_m, \ldots, A_0 \) are not all “effective” unknowns. The term \( A_0 \) \( e^{\alpha t} \) does not survive the homogeneous equation and disappears. So effectively we have \( m \) unknowns \( A_m, \ldots, A_1 \) trying to satisfy \( m + 1 \) relations, which is bound to fail.

The fix? We add one term,

\[
y_p = (A_{m+1} t^{m+1} + A_m t^m + \cdots + A_1 t + A_0) e^{\alpha t}.
\] (182)

As \( A_0 \) is useless anyway, we drop it, to write

\[
y_p = t (A_{m+1} t^{m+1} + \cdots + A_1) e^{\alpha t}.
\] (183)

Same idea applies to double roots.

This way of counting is very easy to extend to higher order equations. Consider

\[
y^{(4)} + 2 y'' + y = 3 \sin t - 5 \cos t.
\] (184)

Turns out the general solution to the homogeneous equation is

\[
y = C_1 \sin t + C_2 \cos t + C_3 t \sin t + C_4 t \cos t.
\] (185)

Now following the form of the RHS we guess

\[
y_p = (A_k t^k + \cdots + A_1 t + A_0) \sin t + (B_k t^k + \cdots + B_1 t + B_0) \cos t.
\] (186)

The RHS contains 2 terms. We need to balance it with two effective unknowns. Clearly \( A_1, A_0, B_1, B_0 \) are not effectively. Therefore the correct guess is

\[
y_p = A t^2 \sin t + B t^2 \cos t.
\] (187)
Armed with the general procedure, we now study a few more examples.

**Example 20.** Find a particular solution to
\[ y''(x) + y(x) = 4x \cos x. \]  
(188)

**Solution.** First we need to figure out the correct form of the solution. Comparing with the general procedure above, we write the right hand side as
\[ 4x e^{0x} \cos x. \]  
(189)
We see that the polynomial involved is \( x \), which is of degree one. This suggests that a factor \( (A_1 x + A_0) \) is in the solution. Thus the solution is of the form
\[ x^s (A_1 x + A_0) \cos x + t^s (B_1 x + B_0) \sin x. \]  
(190)
To fix the power \( s \), we need to check whether \( 0 + i \) is a root to the auxiliary equation
\[ r^2 + 1 = 0. \]  
(191)
Clearly the answer is yes. Therefore we need to take \( s = 1 \).
Substituting
\[ y = x [(A_1 x + A_0) \cos x + (B_1 x + B_0) \sin x] \]  
(192)
into the equation, we have
\[
4x \cos x = y'' + y \\
= \left[ (A_1 x^2 + A_0 x) \cos x + (B_1 x^2 + B_0 x) \sin x \right]' \\
+ (A_1 x^2 + A_0 x) \cos x + (B_1 x^2 + B_0 x) \sin x \\
= \left[ 2A_1 \cos x - 2(2A_1 x + A_0) \sin x - (A_1 x^2 + A_0 x) \cos x \right] \\
+ \left[ 2B_1 \sin x + 2(2B_1 x + B_0) \cos x - (B_1 x^2 + B_0 x) \sin x \right] \\
+ (A_1 x^2 + A_0 x) \cos x + (B_1 x^2 + B_0 x) \sin x \\
= 4B_1 \cos x + (-4A_1) x \sin x + (2A_1 + 2B_0) \cos x + (2B_1 - 2A_0) \sin x.
\]  
(193)
Thus
\[
4B_1 = 4 \\
-4A_1 = 0 \\
2A_1 + 2B_0 = 0 \\
2B_1 - 2A_0 = 0.
\]  
(194) (195) (196) (197)
Solving this gives
\[ B_1 = 1, \ A_1 = 0, \ B_0 = 0, \ A_0 = 1. \]  
(198)
So the particular solution is given by
\[ y_p = x [\cos x + x \sin x]. \]  
(199)

**Example 21.** Find a particular solution to
\[ y'' - 2y' + 3y = \cosh t. \]  
(200)

**Solution.** First recall that
\[ \cosh t = \frac{1}{2} [e^t + e^{-t}]. \]  
(201)
Thus to find the particular solution, we need to find a particular solution for
\[ y'' - 2y' + 3y = \frac{1}{2} e^t \]  and \[ y'' - 2y' + 3y = \frac{1}{2} e^{-t}. \]  
(202)
respectively.

- Particular solution for

\[ y'' - 2y' + 3y = \frac{1}{2} e^t. \]  \hspace{1cm} (203)

Guided by the method of undetermined coefficients, we look for solutions of the form

\[ y = A t^s e^t. \]  \hspace{1cm} (204)

To determine \( s \), we need to check the relation between \( r = 1 \) and the auxiliary equation

\[ r^2 - 2r + 3 = 0. \]  \hspace{1cm} (205)

We see that \( r = 1 \) is not a solution. Thus \( s = 0 \). Substituting \( y = A e^t \) into the equation we reach

\[ \frac{1}{2} e^t = y'' - 2y' + 3y = 2A e^t \implies A = \frac{1}{4} \implies y_{p1} = \frac{1}{4} e^t. \]  \hspace{1cm} (206)

- Particular solution for

\[ y'' - 2y' + 3y = \frac{1}{2} e^{-t}. \]  \hspace{1cm} (207)

We look for solutions of the form

\[ y = A t^s e^{-t}. \]  \hspace{1cm} (208)

This time \( r = -1 \) is also not a solution to the auxiliary equation

\[ r^2 - 2r + 3 = 0 \]  \hspace{1cm} (209)

and consequently \( s = 0 \). Substituting \( y = A e^{-t} \) into the equation leads to

\[ \frac{1}{2} e^{-t} = y'' - 2y' + 3y = 6A e^{-t} \implies A = \frac{1}{12} \implies y_{p2} = \frac{1}{12} e^{-t}. \]  \hspace{1cm} (210)

Putting everything together, the particular solution is given by

\[ y_p = \frac{1}{4} e^t + \frac{1}{12} e^{-t}. \]  \hspace{1cm} (211)

The same idea works for higher order equations.

**Example 22.** Find a particular solution to

\[ y'''' + y'' - 2y = t e^t. \]  \hspace{1cm} (212)

**Solution.** We look for solutions of the form

\[ y = t^s [A_1 t + A_0] e^t. \]  \hspace{1cm} (213)

To determine \( s \) we need to check the relation between \( r = 1 \) and the auxiliary equation

\[ r^3 + r^2 - 2 = 0 \iff (r - 1) (r^2 + 2r + 2) = 0. \]  \hspace{1cm} (214)

We see that \( r = 1 \) is a simple root. Therefore \( s = 1 \). Substituting \( y = t (A_1 t + A_0) e^t \) into the equation we have

\[
  t e^t = y'''' + y'' - 2y \\
  = \left[ (A_1 t^2 + A_0 t) e^t \right]'''' + \left[ (A_1 t^2 + A_0 t) e^t \right]'' - 2 \left( A_1 t^2 + A_0 t \right) e^t \\
  = \left[ A_1 t^2 + (6A_1 + A_0) t + 6A_1 + 3A_0 \right] e^t + \left[ A_1 t^2 + (4A_1 + A_0) t + 2A_1 + 2A_0 \right] e^t \\
  = 10 A_1 t e^t + (8A_1 + 5A_0) e^t.
\]  \hspace{1cm} (215)

Thus

\[ A_1 = \frac{1}{10}, \quad A_0 = -\frac{4}{25}. \]  \hspace{1cm} (216)
The special solution is then given by

\[ y_p(t) = t \left[ \frac{t}{10} - \frac{4}{25} \right] e^t = \left( \frac{t^2}{10} - \frac{4t}{25} \right) e^t. \]  

(217)

3.2. Variation of parameters.

What if the right hand side is not a combination of powers, exponential, sin’s and cos’s? In this section we present a more general method: variation of parameters. This method yields a particular solution for any linear equation as long as all solutions to the corresponding homogeneous equation are already known.

Consider the nonhomogeneous problem

\[ a y'' + b y' + c y = g(t). \]  

(218)

As we can easily solve the homogeneous equation

\[ a y'' + b y' + c y = 0, \]  

(219)

we would like to check the possibility of obtaining the particular solution from modifying the general solutions of the homogeneous problem.

More specifically, let \( y_1, y_2 \) be two linearly independent solutions to the homogeneous problem. We know that the general solution of the homogeneous problem is given by

\[ y = c_1 y_1 + c_2 y_2. \]  

(220)

Now the idea is to replace the constants \( c_1, c_2 \) by two functions and try to find particular solution of the form

\[ y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t). \]  

(221)

Substituting this into the nonhomogeneous equation, we obtain

\[
g(t) = a y'' + b y' + c y \\
= a \left[ v_1 y_1 + v_2 y_2 \right]'' + b \left[ v_1 y_1 + v_2 y_2 \right]' + c \left[ v_1 y_1 + v_2 y_2 \right] \\
= a \left[ v_1'' y_1 + 2 v_1' y_1' + v_1 y_1'' + v_2'' y_2 + 2 v_2' y_2' + v_2 y_2'' \right] \\
+ b \left[ v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' \right] + c \left[ v_1 y_1 + v_2 y_2 \right] \\
= v_1 \left[ a y_1'' + b y_1' + c y_1 \right] + v_2 \left[ a y_2'' + b y_2' + c y_2 \right] \\
+ a \left[ v_1'' y_1 + 2 v_1' y_1' + v_1 y_1'' + v_2'' y_2 + 2 v_2' y_2' + v_2 y_2'' \right] + b \left[ v_1' y_1 + v_2' y_2 \right] \\
= a \left[ v_1'' y_1 + 2 v_1' y_1' + v_2'' y_2 + 2 v_2' y_2' \right] + b \left[ v_1' y_1 + v_2' y_2 \right]. \]  

(222)

Thus as long as we can find \( v_1, v_2 \) such that

\[ a \left[ v_1'' y_1 + 2 v_1' y_1' + v_2'' y_2 + 2 v_2' y_2' \right] + b \left[ v_1' y_1 + v_2' y_2 \right] = g(t), \]  

(223)

we are done.

This equation is quite complicated. We need to simplify it. Note that we have two unknowns \( v_1, v_2 \) and just one equation, it is likely that we can require \( v_1, v_2 \) to satisfy another equation without losing existence of solutions.

One good choice is requiring

\[ v_1' y_1 + v_2' y_2 = 0. \]  

(224)

Note that this not only makes the term containing \( b \) vanish, but also greatly simplify the first term through

\[ 0 = \left( v_1' y_1 + v_2' y_2 \right)' = v_1'' y_1 + v_1 y_1'' + v_2'' y_2 + v_2 y_2''. \]  

(225)

Thus our equations for \( v_1, v_2 \) are

\[
a \left[ v_1' y_1' + v_2' y_2' \right] = g(t) \\
v_1' y_1 + v_2' y_2 = 0. \]  

(226)  

(227)
Rewrite this system to
\begin{align*}
y_1' v_1' + y_2' v_2' &= g(t)/a \\
y_1 v_1' + y_2 v_2' &= 0.
\end{align*}
This system admits a unique solution when \( y_1, y_2 \) are linearly independent.\(^9\) The solution is given by
\begin{align*}
v_1' &= \frac{-g(t) y_2(t)}{a [y_1 y_2' - y_1' y_2]} , \quad v_2' = \frac{g(t) y_1(t)}{a [y_1 y_2' - y_1' y_2]}.
\end{align*}
Then \( v_1, v_2 \) can be obtained from integrating the above.

**Example 23.** Find the general solution to the differential equation
\[ y'' + 4y = \tan 2t. \tag{235} \]

**Solution.** We need to do two things: finding general solution to the homogeneous equation, and finding one particular solution to the nonhomogeneous equation.

- **General solution to the homogeneous problem.** The auxiliary equation is
\[ r^2 + 4 = 0 \implies r_1 = 2i, \quad r_2 = -2i \implies z_1 = e^{2it} = \cos 2t + i \sin 2t, \quad z_2 = \cos 2t - i \sin 2t. \tag{236} \]
Thus the general solution to the homogeneous equation is given by
\[ \hat{y} = c_1 \cos 2t + c_2 \sin 2t. \tag{237} \]

- **Particular solution to the nonhomogeneous problem.** As the right hand side is \( \tan 2t \), it is not possible to use the method of undetermined coefficients. We use variation of parameters instead. We have
\[ y_1 = \cos 2t, \quad y_2 = \sin 2t, \quad g(t) = \tan 2t = \frac{\sin 2t}{\cos 2t}, \quad a = 1. \tag{238} \]
We compute
\[ y_1 y_2' - y_1' y_2 = 2. \tag{239} \]
Thus the equations for \( v_1, v_2 \) are
\begin{align*}
v_1' &= \frac{-g(t) y_2(t)}{2 \cos 2t} = \frac{1}{2} \sin 2t , \quad v_2' = \frac{g(t) y_1(t)}{2 \cos 2t} = \frac{1}{2} \sin 2t.
\end{align*}
Integrating, we have
\begin{align*}
v_1(t) &= \frac{1}{2} \int \frac{\sin 2t}{\cos 2t} \, dt \\
&= \frac{1}{2} \int \frac{1 - \cos 2t}{\cos 2t} \, dt \\
&= \frac{1}{2} \int \frac{1}{\cos 2t} \, dt + \frac{1}{2} \int \cos 2t \, dt. \tag{241}
\end{align*}

---

9. Recall from linear algebra that the system
\begin{align*}
ax + by &= e \\
cx + dy &= f
\end{align*}
has a unique solution for all \( e, f \) if and only if
\[ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0. \tag{232} \]
When this is the case, the unique solution is given by Cramer’s rule
\begin{align*}
x &= \frac{\det \begin{pmatrix} e & b \\ f & d \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} , \quad y = \frac{\det \begin{pmatrix} a & e \\ c & f \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}. \tag{233}
\end{align*}
We evaluate
\[ \frac{1}{2} \int \cos 2t \, dt = \frac{1}{4} \sin 2t. \tag{242} \]
\[- \frac{1}{2} \int \frac{1}{\cos 2t} \, dt = - \frac{1}{2} \int \frac{\cos 2t}{\cos^2 2t} \, dt \]
\[= - \frac{1}{4} \int \frac{d\sin 2t}{1 - \sin^2 2t} \]
\[= - \frac{1}{8} \left[ \int \frac{d\sin 2t}{1 - \sin 2t} + \int \frac{d\sin 2t}{1 + \sin 2t} \right] \]
\[= - \frac{1}{8} \left[ \ln |1 + \sin 2t| - \ln |1 - \sin 2t| \right]. \tag{243} \]
Thus
\[ v_1(t) = \int \frac{\sin^2 2t}{\cos 2t} \, dt = - \frac{1}{8} [\ln |1 + \sin 2t| - \ln |1 - \sin 2t|] + \frac{1}{4} \sin 2t. \tag{244} \]
On the other hand,
\[ v_2(t) = \frac{1}{2} \int \sin 2t \, dt = - \frac{1}{4} \cos 2t. \tag{245} \]
Putting things together, we have
\[ y_p(t) = v_1 y_1 + v_2 y_2 \]
\[= \left[ - \frac{1}{8} [\ln |1 + \sin 2t| - \ln |1 - \sin 2t|] + \frac{1}{4} \sin 2t \right] \cos 2t \]
\[- \frac{1}{4} \cos 2t \sin 2t \]
\[= - \frac{1}{8} [\ln |1 + \sin 2t| - \ln |1 - \sin 2t|] \cos 2t. \tag{246} \]
Thus the desired general solution is given by
\[ y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{8} [\ln |1 + \sin 2t| - \ln |1 - \sin 2t|] \cos 2t. \tag{247} \]

Example 24. Solve
\[2 y'' - 2 y' - 4 y = 2 e^{3t}. \tag{248} \]

Solution. Clearly we should first simplify the equation to
\[ y'' - y' - 2 y = e^{3t}. \tag{249} \]
- General solution to
\[ y'' - y' - 2 y = 0. \tag{250} \]
The auxiliary equation is
\[ r^2 - r - 2 = 0 \implies r_1 = 2, \quad r_2 = -1. \tag{251} \]
Thus the general solution is given by
\[ y = c_1 e^{2t} + c_2 e^{-t}. \tag{252} \]
- Finding one particular solution to
\[ y'' - y' - 2 y = e^{3t}. \tag{253} \]
We use variation of parameters. We have
\[ y_1 = e^{2t}, \quad y_2 = e^{-t}, \quad g = e^{3t}, \quad a = 1. \tag{254} \]
We compute
\[ y_1 y_2' - y_1' y_2 = - e^t - 2 e^t = -3 e^t. \tag{255} \]
Thus we have

\[ v_1' = \frac{-g(t) y_2(t)}{a [y_1 y_2 - y_1' y_2]} = \frac{1}{3} e^t, \quad v_2 = \frac{g(t) y_1(t)}{a [y_1 y_2 - y_1' y_2]} = -\frac{1}{3} e^{4t}. \]  

(256)

Integrating, we have

\[ v_1(t) = \frac{1}{3} e^t, \quad v_2(t) = -\frac{1}{12} e^{4t}. \]

(257)

The particular solution is then given by

\[ y_p(t) = v_1 y_1 + v_2 y_2 = \frac{1}{4} e^{3t}. \]

(258)

Finally, the general solution to the nonhomogeneous problem is

\[ x(t) = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{4} e^{3t}. \]

(259)

Remark 25. Clearly we can also find the particular solution to

\[ x'' - x' - 2x = e^{3t}. \]  

(260)

using the method of undetermined coefficients: Guess \( x = Ae^{3t} \) and substitute into the equation:

\[ [9 A - 3 A - 2 A] e^{3t} = e^{3t} \Rightarrow A = \frac{1}{4} \Rightarrow x_p = \frac{1}{4} e^{3t}. \]

(261)

Remark 26. As we can see from the above calculation, when the method of undetermined coefficients can be applied, usually it’s a bit simpler than the method of variation of parameters. On the other hand, variation of parameters can be applied to any right hand side. More importantly, it’s theoretically much cleaner and can be generalized to the case of variable-coefficient equations.

<table>
<thead>
<tr>
<th>Undetermined Coefficients</th>
<th>Variation of Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applicable to</td>
<td>Some ( g )</td>
</tr>
<tr>
<td>Need to solve ( a y'' + b y' + c y = 0 )?</td>
<td>Sometimes</td>
</tr>
<tr>
<td>Complexity of calculation</td>
<td>Low</td>
</tr>
<tr>
<td>Extension to variable coefficient equations?</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 1. Solving \( a y'' + b y' + c y = g(t) \), UC vs. VP.

Remark 27. A few more remarks.

1. Although variation of parameters can be extended to variable coefficient equations, in practice it is rarely used for the following reasons:
   - There is no good method getting formulas for solutions \( y_1, y_2 \) of the homogeneous equation \( a(t) y'' + b(t) y' + c(t) y = 0 \). Such formulas may not even exist;
   - The “series” method can get \( y_1, y_2 \) in the form of infinite sums. But they are hard to use in variation of parameters. Furthermore the same “series” method can get \( y_p \) directly without any more difficulty than getting \( y_1, y_2 \).

2. Back to constant coefficient equations. We will introduce a third method, the method of Laplace transform, which is in some sense in between the current two methods: Simpler calculation than variation of parameters, but more complicated than undetermined coefficients, etc.

4. Variable-coefficient equations.

   The theory is still true: For an \( n \)th order linear equation

\[ a_n(t) y^{(n)} + \cdots + a_0(t) y = g(t), \]  

(262)

10. The higher the order, the simpler (compared to variation of parameters).
the following are true on intervals over which \(a_i/a_n, i = 0, \ldots, n-1\), are simultaneously continuous:

- if \(g(t) = 0\) (homogeneous), the general solution is
  \[
  c_1 y_1 + \cdots + c_n y_n
  \]
  where \(y_1, \ldots, y_n\) are linearly independent solutions;
- Otherwise, the general solution is
  \[
  c_1 y_1 + \cdots + c_n y_n + y_p
  \]
  where \(y_p\) is a particular solution and \(y_1, \ldots, y_n\) are as above.
- Linear dependence/independence can still be checked using Wronskian.
- The method of variation of parameters can be generalized to the variable-coefficient case almost without any modification. For example, consider the case of second order equation
  \[
  a(t) y'' + b(t) y' + c(t) y = g(t).
  \]
  Let \(y_1, y_2\) be two linearly independent solutions to the homogeneous equation, then \(v_1 y_1 + v_2 y_2\) is a particular solution, where
  \[
  v_1' = \frac{-g y_2}{a W[y_1, y_2]}, \quad v_2' = \frac{g y_1}{a W[y_1, y_2]}.
  \]

However, finding \(y_1, \ldots, y_n\) becomes much much harder. We have to restrict ourselves in special cases or rely on luck (combined with educated guesses whose accuracy depends on experience). Furthermore, for any important variable coefficient equations, the solutions cannot be expressed via classical functions (polynomial, trigonometric, exponential, logarithm, etc.).

**Example 28. (Cauchy-Euler equation)** Solve
\[
t^2 y'' + 7 t y' - 7 y = 0.
\]

**Solution.** Guided by the theory, we only need to find two linearly independent solutions. The key now is to realize the following property of \(t\): \((t^r)^{(k)} t^k = C t^r\). Substitute \(y = t^r\) into the equation, we have
\[
0 = t^2 y'' + 7 t y' - 7 y = [r (r - 1) + 7 r - 7] t^r = 0 \iff r^2 + 6 r - 7 = 0 \implies r_1 = -7, r_2 = 1.
\]
Thus the general solution is
\[
y = c_1 t^{-7} + c_2 t.
\]

One easily sees that this strategy would work for equations of the form
\[
a t^2 y'' + b t y' + c y = 0
\]
where \(a, b, c\) are constants. Such equations, together with the nonhomogeneous version
\[
a t^2 y'' + b t y' + c y = g(t)
\]
are called Cauchy-Euler, or Equidimensional, equations.

Consider the homogeneous Cauchy-Euler equation
\[
a t^2 y'' + b t y' + c y = 0
\]
Substituting \(y = t^r\) leads to
\[
ar (r - 1) + br + c = 0.
\]
This is called the associated characteristic equation.

As the characteristic equation is quadratic, we have three cases:
- Two distinct real roots \(r_1, r_2\). In this case the two linearly independent solutions are
  \[
t^{r_1}, \quad t^{r_2}.
\]
A double root \( r = r_1 = r_2 \). As \( a, b, c \) are real, \( r \) is real. In this case \( t^r \) is one solution and we need another one. It turns out we can take the second one to be \( y_2 = t^r \ln t \). (275)

A pair of complex roots \( r_{1,2} = \alpha \pm i \beta \). Then we have
\[
t^{r_1} = e^{(\alpha + i\beta)\ln t} = e^{\alpha \ln t} [\cos (\beta \ln t) + i \sin (\beta \ln t)], \quad t^{r_2} = e^{\alpha \ln t} [\cos (\beta \ln t) - \sin (\beta \ln t)].
\] (276)

Using the linearity of the solution, we see that two linearly independent solutions are given by
\[
y_1 = e^{\alpha \ln t} \cos (\beta \ln t), \quad y_2 = e^{\alpha \ln t} \sin (\beta \ln t).
\] (277)

**Remark 29.** The above looks similar to our theory for linear constant-coefficient equations. This similarity becomes more striking if we introduce a new variable \( x = \ln t \): The three cases become

- Two distinct roots \( \Longrightarrow e^{\alpha x}, e^{\beta x} \);
- One double root \( \Longrightarrow e^{\alpha x}, xe^{\alpha x} \);
- Complex roots \( \Longrightarrow e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x \).

This is no coincidence! In fact, setting \( x = \ln t \) gives
\[
y' = \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = t^{-1} \frac{dy}{dx}, \quad y' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ t^{-1} \frac{dy}{dx} \right] \frac{dx}{dt} = t^{-2} \frac{d^2y}{dx^2} - t^{-2} \frac{dy}{dx}.
\] (278)

Substituting into the equation
\[
0 = a t^2 y'' + b t y' + c y = a \frac{d^2y}{dx^2} + (b-a) \frac{dy}{dx} + c y.
\] (279)

Thus we have transformed the Euler-Cauchy equation into a constant-coefficient equation. Furthermore, the auxiliary equation for this equation is
\[
a r^2 + (b-a) r + c = a r (r-1) + br + c
\] (280)

which is exactly the characteristic equation of the Cauchy-Euler equation!

**Example 30.** Solve the following initial value problem for the Cauchy-Euler equation
\[
t^2 y'' - 4 t y' + 4 y = 0; \quad y(1) = -2, \quad y'(1) = -11.
\] (281)

**Solution.** Substituting \( y = t^r \) gives
\[
r (r-1) - 4 r + 4 = 0 \iff r^2 - 5 r + 4 = 0 \iff r_1 = 4, \quad r_2 = 1.
\] (282)

Thus the general solution is given by
\[
y(t) = c_1 t^4 + c_2 t.
\] (283)

Using the initial values, we have
\[
-2 = y(1) = c_1 + c_2; \quad -11 = y'(1) = 4 c_1 + c_2.
\] (284)

Solving this we reach
\[
c_1 = -3, \quad c_2 = 1.
\] (285)

Thus the solution to the initial value problem is given by
\[
y(t) = -3 t^4 + t.
\] (286)

**Example 31.** Solve
\[
t^2 y'' - 3 t y' + 4 y = 0.
\] (287)

**Solution.** Substituting \( y = t^r \) we have
\[
r (r-1) - 3 r + 4 = 0 \iff r^2 - 4 r + 4 = 0 \iff r_1 = r_2 = 2.
\] (288)
This is double root so the general solution is given by
\[ y = c_1 t^2 + c_2 t^2 \ln t. \] (289)

**Example 32.** Solve
\[ y'' - \frac{1}{t} y' + \frac{5}{t^2} y = 0. \] (290)

**Solution.** Multiply both sides by \( t^2 \):
\[ t^2 y'' - t y' + 5 y = 0. \] (291)
Substituting \( y = t^r \) gives
\[ r (r - 1) - r + 5 = 0 \iff r^2 - 2r + 5 = 0 \iff r_1 = 1 + 2i, r_2 = 1 - 2i. \] (292)
The general solution is then given by
\[ y = c_1 t \cos (2 \ln t) + c_2 t \sin(2 \ln t). \] (293)

Cauchy-Euler equation is just one special class of equations. For most equations, however, we will not be able to obtain two linearly independent solutions from guessing. On the other hand, we may be able to find one single solution via guesswork. Fortunately, we can use this one solution as leverage and pull the following trick to obtain a 2nd solution which is linearly independent of the guessed one, and thus obtain the general solution.

Consider the equation
\[ y'' + p(t) y' + q(t) y = 0. \] (294)
Let \( y_1 \) be a solution. The idea is to find \( v(t) \) such that \( y_2 = v y_1 \) is also a solution. Note that this clearly requires \( y_1 \neq 0 \), since otherwise \( y_2 = y_1 = 0 \) no matter what \( v \) is. On the other hand, as long as \( v \) is not a constant, \( y_1 \) and \( y_2 \) are linearly independent.\(^{11}\)

Substituting \( y = v y_1 \) into the equation we obtain
\[
0 = y'' + p(t) y' + q(t) y = (v y_1)'' + p(t) (v y_1)' + q(t) v y_1 = v y_1'' + 2 v' y_1' + v'' y_1 + p v y_1' + p v' y_1 + q v y_1 = v [y_1'' + p y_1' + q y_1] + v'' y_1 + 2 v' y_1 + p v' y_1 = v'' y_1 + (2 y_1' + p y_1) v'.
\] (295)
Thus all we need to do is to solve
\[ y_1 v'' + (2 y_1' + p y_1) v' = 0. \] (296)
This is a linear first order equation (after setting \( w = v' \)). Thus we should multiply both sides by
\[ e^{\int (2 y_1' + p y_1)/y_1} = e^{\frac{2}{y_1} \int p} e^{\int p} = y_1^2 e^{\int p} \] (297)
to obtain
\[
(y_1^2 e^{\int p} v')' = 0 \implies y_1^2 e^{\int p} v' = C \implies v' = \frac{C e^{-\int p}}{y_1^2} \implies v = C \int \frac{e^{-\int p(t) dt}}{y_1(t)^2} dt.
\] (298)
As all we need is \( v \) satisfying
\[ y_1 v'' + (2 y_1' + p y_1) v' = 0. \] (299)
we have the freedom to take any constant \( C \). For example, we can take \( C = 1 \) to obtain the following theorem.

**Theorem 33. (Reduction of order)** Consider the equation
\[ y'' + p(t) y' + q(t) y = 0. \] (300)

---

\(^{11}\) Another linear algebra exercise!
Let $y_1$ be a solution, not identically zero. Then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) \, dt}}{y_1(t)^2} \, dt$$

(301)

is a second, linearly independent solution.

**Example 34.** Solve

$$ty'' - (t + 1) \, y' + y = t^2, \quad t > 0.$$  

(302)

**Solution.** Based on the above theory, we proceed as follows.

1. Find the general solution to the homogeneous equation

$$ty'' - (t + 1) \, y' + y = 0, \quad t > 0.$$  

(303)

This step consists of two substeps:

a. Find one nonzero solution through guessing.

b. Find a second one via reduction of order.

2. Find a particular solution to the nonhomogeneous equation using variation of parameters.

Now we carry this plan out.

1. Find the general solution to the homogeneous equation

$$ty'' - (t + 1) \, y' + y = 0, \quad t > 0.$$  

(304)

a. Find one nonzero solution through guessing. We notice that $t - (t + 1) + 1 = 0$. Thus if we can find $y$ such that $y'' = y'$, this $y$ is a solution. Fortunately such $y$ exists: $y = e^t$.

b. Find a second one via reduction of order. We have found $y_1 = e^t$. Rewrite the equation into

$$y'' - \frac{t + 1}{t} \, y' + \frac{1}{t} \, y = 0.$$  

(305)

Thus

$$p(t) = -\left(1 + \frac{1}{t}\right), \quad q(t) = \frac{1}{t}.$$  

(306)

Calculate

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) \, dt}}{y_1(t)^2} \, dt$$

$$= e^t \int \frac{e^{-\int (1+1/t) \, dt}}{e^{2t}}$$

$$= e^t \int \frac{e^{t} e^{\ln t}}{e^{2t}}$$

(307)

(307)

(Note that as $t > 0, |t| = t$)

$$= e^t \int t \, e^{-t} \, dt$$

$$= e^t \left[ -t \, e^{-t} + \int e^{-t} \, dt \right]$$

$$= e^t \left[ -t \, e^{-t} - e^{-t} \right]$$

$$= - (t + 1).$$

As the above calculation is quite complicated, it is a good idea to substitute $y = - (t + 1)$ into the equation and see that it is indeed a solution.

As the equation is linear, $-c \,(t + 1)$ can also serve as $y_2$ for arbitrary constant $c \neq 0$. We take $c = -1$ to make things simple.

Thus the general solution to the homogeneous equation is given by

$$y = c_1 \, e^t + c_2 \,(t + 1).$$  

(308)
2. Now we apply variation of parameters, that is, we try to find a particular solution to the nonhomogeneous equation

\[ t y'' - (t + 1) y' + y = t^2, \quad t > 0. \]  

of the form

\[ y = v_1 y_1 + v_2 y_2. \]

Here \( y_1 = e^t \), \( y_2 = (t + 1) \).

Compute the Wronskian

\[ W[y_1, y_2](t) = y_1 y'_2 - y'_1 y_2 = e^t - e^t(t + 1) = -t e^t. \]

Then we have (Here \( a_2 = t, g = t^2 \))

\[ v_1' = -\frac{g y_2}{a_2 W[y_1, y_2]} = e^{-t}(t + 1), \quad v_2' = \frac{g y_1}{a_2 W[y_1, y_2]} = -1. \]

Integrating, we obtain

\[ v_1 = \int (t + 1) e^{-t} \, dt = - \int (t + 1) de^{-t} = -(t + 1)e^{-t} + \int e^{-t} \, d(t + 1) = -(t + 1)e^{-t} + \int e^{-t} \, dt = -(t + 2)e^{-t}. \]

\[ v_2 = \int -1 = -t. \]

Thus the particular solution is given by

\[ y_p = v_1 y_1 + v_2 y_2 = -(t + 2) + (-t)(t + 1) = -t^2 - 2(t + 1). \]

As \((t + 1)\) is a solution to the homogeneous equation, we can simply take \( y_p = -t^2 \).

Summarizing, we have the general solution to be

\[ y = c_1 e^t + c_2 (t + 1) - t^2. \]

Note that even if we have used \( y_p = -t^2 - 2(t + 1) \), the term \(-2(t + 1)\) would be absorbed into \( c_2 (t + 1) \) here.

**Remark 35.** (Thanks to Mr. Boblin Travis for showing me this) One can also start from \( y_1 = (t+1) \). In that case we need to solve a formidable looking integral:

\[ v = \int \frac{t e^t}{(t+1)^2} \, dt = -\int t e^t \, d\frac{1}{t+1} = -\frac{t e^t}{t+1} + \int \frac{1}{t+1} (e^t \, dt + t e^t \, dt) = -\frac{t e^t}{t+1} + \int e^t = e^t. \]

Thus we obtain \( y_2 = v y_1 = e^t. \)
Remark 36. We see some interesting symmetry here. If we start from $y_1 = e^t$, we obtain $y_2 = -(t + 1)$. If we start from $y_1 = t + 1$, we reach $y_2 = e^t$. This is not a coincidence. Indeed, suppose we start from a solution $y_1$, then use reduction of order to get $y_2 = v y_1$. Now if we instead start from $y_2$ and use reduction of order, we will always get $-y_1$. To see this, we investigate the formula

$$y_2 = y_1 v = y_1 \int \frac{e^{-f_p}}{y_1^2}.$$  \hfill (318)

Divide both sides by $y_1$ and then differentiate, we reach

$$\left( \frac{y_2}{y_1} \right)' = e^{-f_p} y_1' y_1 - y_2 y_1' = e^{-f_p}.$$  \hfill (319)

On the other hand, if we start from $y_2$ and use reduction of order, we would have

$$\tilde{y}_1 = y_2 \int e^{-f_p} y_1^2$$  \hfill (320)

which is equivalent to

$$\tilde{y}_1' y_2 - y_2' \tilde{y}_1 = e^{-f_p}.$$  \hfill (321)

Thus $y_1$ and $\tilde{y}_1$ are related by the equation

$$\tilde{y}_1' y_2 - y_2' \tilde{y}_1 = -(y_1' y_2 - y_2' y_1).$$  \hfill (322)

Assuming the solution is unique, we see that it has to be $\tilde{y}_1 = -y_1$.

Remark 37. Let $y_1$ be a given solution. Let $y$ be any other solution. Let $v = y/y_1$. Then $v$ satisfies

$$v'' + \left( \frac{2 y_1'}{y_1} + p \right) v' = 0.$$  \hfill (323)

The general solution to this equation is

$$v = C_2 \int e^{-f_p} + C_1.$$  \hfill (324)

Then we have

$$y = C_2 y_1 \int e^{-f_p} y_1^2 + C_1 y_1 = C_1 y_1 + C_2 y_2$$  \hfill (325)

where

$$y_2 = y_1 \int e^{-f_p}.$$  \hfill (326)

is the solution we obtained through the reduction of order formula.