System of First Order Linear Equations

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# 1. A Quick Tour via An Example.

## 1.1. Introduction.

Right after the invention of calculus, differential equations replaced algebraic equations (which in turn replaced counting) as the major tool in mathematically modeling everything. A single differential equation (also called "scalar differential equation") is a mathematical model of the time-evolution/spatial variation of one single substance (can be population of a single species, amount of a single chemical, etc.); On the other hand, a system of differential equations models the time-evolution of more than one quantities. One example is Newton's second law:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x} = m\,\boldsymbol{a} \tag{1}$$

which looks like a single equation but is actually a system because both $\boldsymbol{x}$ and $\boldsymbol{a}$ has more than one components. Traditionally, systems of ordinary differential equations arise from study of mechanics. Modern examples also abound, especially from biology, sociology, economics, etc.

The general form of a system involving $n$ unknown functions is

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) && (2)\\ \dot{x}_2 &= f_2(x_1, \ldots, x_n) && (3)\\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) && (4) \end{aligned}$$

where the evolution of $n$ quantities are described. Such a system is usually referred to as an $n \times n$ first order system.

**Remark 1.** When $n = 2$ or 3, $x, y$ (respectively $x, y, z$) are often used instead of $x_1, \ldots, x_n$.

When all $f_1$, ..., $f_n$ are linear in their variables $x_1$, ..., $x_n$, the system is called linear, otherwise it's called nonlinear. So an $n \times n$ first order linear system has the general form

$$\begin{aligned} \dot{x}_1 &= a_{11}(t)\,x_1 + \cdots + a_{1n}(t)\,x_n + g_1(t) && (5)\\ &\vdots \\ \dot{x}_n &= a_{n1}(t)\,x_1 + \cdots + a_{nn}(t)\,x_n + g_n(t). && (6) \end{aligned}$$

If furthermore all $a_{ij}(t)$ are constants, that is

$$\dot{x}_1 = a_{11}x_1 + \cdots + a_{1n}x_n + g_1(t) \tag{7}$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + \cdots + a_{nn}x_n + g_n(t). \tag{8}$$

The system is said to have "constant coefficients". As usual, when $g_1(t) \equiv \cdots \equiv g_n(t) = 0$, the above linear systems are called "homogeneous".

**Remark 2.** In almost all practical cases, the first order system will be nonlinear. There is no systematic way to solve all general nonlinear system. In fact, even for $n \times n$ first order linear system, no simple formula exists (of course unless $n = 1$, which can be solved through application of appropriate integrating factors). Only linear systems with constant coefficients enjoy good formulas for solutions.

Nevertheless, as we will see soon, one important way to understand the general nonlinear system is to derive from it one or more related linear, constant-coefficient systems. Once a good understanding is reached for these constant-coefficient systems, the behaviors of the solutions to the original nonlinear problem often can be obtained.

**Note.** As this is a quick introduction, we will avoid using matrix theory. As a consequence, we will not be able to deal efficiently with systems bigger than $2 \times 2$. We will focus on one particular $2 \times 2$ system, introduced below.

**1.2. The predator-prey model.**

Consider two species of animals. Let $x(t)$, $y(t)$ denote their population. The changes in population size with time can then be modeled by the $2 \times 2$ system

$$\dot{x} = f(x, y)\, x \tag{9}$$

$$\dot{y} = g(x, y)\, y. \tag{10}$$

where $f(x, y)$ and $g(x, y)$ are the "rates of change".

Now let's consider the simplest possible case. First imagine that there is no $y$. Then the most naïve evolution model for $x$ is

$$\dot{x} = a\, x \tag{11}$$

where $a$ is a constant.

Next consider adding the effect of $y$. The simplest model for the effect of $y$ on $x$ is $b\, x\, y$ where $b$ is a constante rate, and $x\, y$ models the "interaction" between two species.

With these considerations we reach the following $2 \times 2$ system:

$$\dot{x} = a\, x + b\, x\, y = (a + b\, y)\, x \tag{12}$$

$$\dot{y} = c\, y + d\, x\, y = (c + d\, x)\, y. \tag{13}$$

As we will see, different natural phenomena correspond to different signs of $a, b, c, d$.

**Example 3. (Competitive Hunters)** In this scenario both $x, y$ are predators hunting on some common preys. Thus they compete with each other.

To model this case, we notice:

- If only one species is present, its population will grow. This means $a > 0, c > 0$.
- The effect of one species on the other is negative. This means $b < 0, d < 0$.

**Example 4. (Predator-Prey)** In this scenario $x$ is the population of the prey while $y$ is the population of the predator. We notice:

- If there is no predator, the population of the prey should grow. Therefore $a > 0$;
- If there is no prey, the population of the predator should decrease. Therefore $c < 0$;
- The effect of the predator on the prey is negative: $b < 0$;

- The effect of the prey on the predator is positive: $d>0$.

We will study the predator-prey model in more detail in the following. For our own convenience, we use a different set of letters to denote various rates:

$$\begin{aligned} \dot{x} &= (b-p\,y)\,x \qquad (14)\\ \dot{y} &= (r\,x-d)\,y. \qquad (15)\end{aligned}$$

Note that all four constants $b,p,r,d$ are positive.

This is a classical $2\times 2$ first order nonlinear system known as Lotka-Volterra system. It is said that Lotka (or Volterra, can't remember)'s son-in-law is the manager of a pond and their after-dinner chats lead to the above model.

**1.3. Dynamics in one dimension.**

The Lotka-Volterra system can be integrated[1]. However as we are trying to understand the general method, we pretend that it cannot.

For systems of nonlinear equations, usually we will not be able to solve them quantitatively, in particular we won't be able to get explicit formulas. As a consequence, "solving" a system is broken into two parts:

- (taks for human) Understand the rough dynamics of the solutions. In particular, if we plot the curves $(x(t), y(t))$ in the plane[2], what would these curves look like? Note that the following important information is lost: How do $x,y$ depend on $t$.
- (task for human+computer) Find to high precision the quantitative relations between $x,y,$ and $t$.

We won't have time to discuss the second task. For the first task, the general approach is so-called "linearization".[3]

To quickly see how the above method works, let's re-consider the scalar case. For example,

$$\dot{x}=(x-1)\,(3-x). \qquad (17)$$

By now of course we know that this equation is "separable" and can therefore be solved explicitly. We also know that the rough behavior of solutions can be obtained through the following steps (without solving the equations):

1. Find constant solutions: $\dot{x}=0 \Longrightarrow x=1$, $x=3$. These constant solutions divide the $t$-$x$ plane into three parts;
2. Discuss the sign of $\dot{x}$ in each part and obtain the asymptotic behavior of solutions.

The "linearization" method is in spirit similar to the second approach, but with an important difference: We will make the $t$-dependence implicit. More specifically, we will just look at the $x$-axis, and treat the equation as describing the movement of a particle: a solution $x(t)$ means at time $t$ the particle is at the position $x(t)$.

Now we first find out those positions where the particle is stationary – once it gets there, it will stay there. Clearly this is characterized by $\dot{x}=0$, or

$$(x-1)\,(3-x)=0. \qquad (18)$$

Thus we identified the equilibrium points (constant solutions): $x=1$ and $x=3$.

1. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\dot{y}}{\dot{x}}=\frac{(r\,x-d)\,y}{(b-p\,y)\,x} \qquad (16)$$

which, although looks ugly, is actually separable.

2. So-called "phase plane".

3. Note that for this particular problem at our hand, after we have written it into the $\frac{\mathrm{d}y}{\mathrm{d}x}=\cdots$ form, we can plot the solution in the phase plane using methods discussed earlier, thus by-passing linearization. But the point here is, linearization still works for bigger systems.

Next we linearize around them. The idea is as follows. We have known that if the particle starts at $x=1$ or 3, it will stay there forever. Now what if the particle starts not exactly at 1 or 3, but very close to them? Can we say anything without solving the full nonlinear equation?

- Close to $x=1$. Set $x=1+X$ where $X$ is small. Substituting into the equation we have

$$\dot{X} = X\,(2-X) = 2\,X - X^2 \approx 2\,X. \tag{19}$$

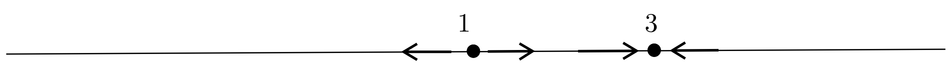
  Thus we see that if $X>0$, $X$ gets more positive; If $X<0$, $X$ gets more negative. Therefore particles start close to $x=1$ are moving away from it. In mathematical language, the equilibrium $x=1$ is "unstable" – although a particle can stay at $x=1$ forever, once it is pushed a little bit away, it will run away.

- Close to $x=3$. Set $x=3+X$ where $X$ is small. Substituting into the equation we have

$$\dot{X} = (X+2)\,(-X) = -2\,X - X^2 \approx -2\,X. \tag{20}$$

  We see that the behavior is the opposite, particles starting from close to $x=3$ get even closer at later time. $x=3$ is called a "stable" equilibrium.

Now we get the following rough picture of the dynamics:




From this picture one naturally guess that the dynamics should be:

- Particles start at $x=1$ and $x=3$ stay put;
- Particles start to the left of $x=1$ moves toward $-\infty$;
- Particles start to the right of $x=1$ (except for those starts at $x=3$) moves toward $x=3$.

Fortunately, existence and uniqueness theory guarantees that the above guess is indeed true.

**1.4. Linearization of the predator-prey model.**

Now back to the Lotka-Volterra Model:

$$\dot{x} = (b - p\,y)\,x \tag{21}$$
$$\dot{y} = (r\,x - d)\,y. \tag{22}$$

Let's "linearize" it.

1. First we have to identify equilibrium points. Setting $\dot{x}=\dot{y}=0$ we reach

$$(b-p\,y)\,x = 0 \tag{23}$$
$$(r\,x-d)\,y = 0 \tag{24}$$

   There are two solutions: $(0,0)$, $(d/r, b/p)$.

2. Linearize around equilibria.

   - Around $(0,0)$. We write $x=0+X$, $y=0+Y$ where $X, Y$ are very small. Substituting into the equations, we have

$$\dot{X} = (b-p\,Y)\,X = b\,X - p\,X\,Y \approx b\,X. \tag{25}$$
$$\dot{Y} = (r\,X-d)\,Y = r\,X\,Y - d\,Y \approx -d\,Y. \tag{26}$$

   - Around $(d/r, b/p)$. We write $x=d/r+X$, $y=b/p+Y$ where $X, Y$ are very small. Substituting into the equations, we have

$$\dot{X} = (-p\,Y)\,(d/r+X) = -\frac{d\,p}{r}\,Y - p\,X\,Y \approx -\frac{d\,p}{r}\,Y; \tag{27}$$
$$\dot{Y} = (r\,X)\,(b/p+Y) = \frac{b\,r}{p}\,X + r\,X\,Y \approx \frac{b\,r}{p}\,X. \tag{28}$$

3. Analyze the dynamics around equilibria. First note that, for the model to make sense, we need all parameters to be positive.

   - Around $(0,0)$, we need to solve the system

     $$\dot{X} = b\,X; \qquad \dot{Y} = -\,d\,Y. \tag{29}$$

     It is easy as the system is de-coupled. The solution is

     $$X = X_0\,e^{bt}, \qquad Y = Y_0\,e^{-dt}. \tag{30}$$

     We can also see the rough dynamics without solving the equations: If we start along $Y$ axis, we will move toward the origin; If we start along $X$ axis, we move away.

   - Around $(d/r, b/p)$ the situation is more complicated. We have

     $$\dot{X} = -\frac{d\,p}{r}\,Y; \qquad \dot{Y} = \frac{b\,r}{p}\,X. \tag{31}$$

     Not de-coupled anymore. But we still can solve it by writing it into one single second order equation. Another way is to plot the behavior of $\left(\dot{X}, \dot{Y}\right)$ in each quadrant.

4. Assemble things together. Here the situation is much more complicated than the 1D case. Consult books on dynamical systems if interested.

## 2. Solving First Order Linear Systems With Constant Coefficients.

From the above example we see that it is crucial to be able to solve general $2 \times 2$ constant-coefficient systems.

$$\begin{aligned} \dot{x} &= a\,x + b\,y && (32)\\ \dot{y} &= c\,x + d\,y. && (33) \end{aligned}$$

with general initial values

$$x(0) = x_0; \qquad y(0) = y_0. \tag{34}$$

### 2.1. What do the solutions look like?.

But how should we solve it? Let's work through an example.

**Example 5.** Solve the $2 \times 2$ system

$$\begin{aligned} \dot{x} &= 3\,x + 2\,y && (35)\\ \dot{y} &= x + 4\,y. && (36) \end{aligned}$$

with $x(0) = 3$, $y(0) = 2$.

Recall what we have learned so far:

- For first order scalar equations: Separation of variables; Integrating factors; Transformations;
- For constant-coefficient equations: Apply formulas; Laplace transform;
- For variable-coefficient equations: Series method.

It turns out that many of these can actually be applied here.

**Transform to scalar equation.**

We write $x = -\,4\,y + \dot{y}$ and substitute into the first equation:

$$(-\,4\,y + y')' = 3\,(-\,4\,y + y') + 2\,y \Longleftrightarrow y'' - 7\,y' + 10\,y = 0. \tag{37}$$

**Series method.**

Write $x = \sum_{n=0}^{\infty} a_n t^n$, $y = \sum_{n=0}^{\infty} b_n t^n$. Substitute into the equations, we have

$$\sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, t^n = \sum_{n=1}^{\infty} n\, a_n\, t^{n-1} = \sum_{n=0}^{\infty} (3\, a_n + 2\, b_n)\, t^n \tag{38}$$

$$\sum_{n=0}^{\infty} (n+1)\, b_{n+1}\, t^n = \sum_{n=1}^{\infty} n\, b_n\, t^{n-1} = \sum_{n=0}^{\infty} (a_n + 4\, b_n)\, t^n. \tag{39}$$

We obtain the recurrence relation

$$a_{n+1} = \frac{3\, a_n + 2\, b_n}{n+1}; \qquad b_{n+1} = \frac{a_n + 4\, b_n}{n+1}. \tag{40}$$

It's not clear how to solve $a_n, b_n$.[4]

**Laplace transform.**

Transforming the equation, we obtain

$$s\, X = 3\, X + 2\, Y + 3 \tag{41}$$

$$s\, Y = X + 4\, Y + 2 \tag{42}$$

This leads to

$$(s-3)\, X - 2\, Y = 3 \tag{43}$$

$$-X + (s-4)\, Y = 2 \tag{44}$$

Solving for $X, Y$ using Cramer's rule we obtain

$$X = \frac{\det\begin{pmatrix} 3 & -2 \\ 2 & s-4 \end{pmatrix}}{\det\begin{pmatrix} s-3 & -2 \\ -1 & s-4 \end{pmatrix}} = \frac{3\, s}{s^2 - 7\, s + 10}; \quad Y = \frac{\det\begin{pmatrix} s-3 & 3 \\ -1 & 2 \end{pmatrix}}{\det\begin{pmatrix} s-3 & -2 \\ -1 & s-4 \end{pmatrix}} = \frac{2\, s - 3}{s^2 - 7\, s + 10}. \tag{45}$$

Taking inverse Laplace transform, we get the solution.

**Remark 6.** If our goal is to find the solution to this particular problem, the first method works best; If our goal is to develop a general theory which will finally include the variable coefficient case, the series method approach is in fact more revealing.

### 2.2. What can happen in the general case.

Now consider the general case:

$$\dot{x} = a\, x + b\, y \tag{46}$$

$$\dot{y} = c\, x + d\, y. \tag{47}$$

with general initial values

$$x(0) = x_0; \qquad y(0) = y_0. \tag{48}$$

Taking Laplace transform we have

$$s\, X = a\, X + b\, Y + x_0 \tag{49}$$

$$s\, Y = c\, X + d\, Y + y_0 \tag{50}$$

which gives

$$(s-a)\, X - b\, Y = x_0 \tag{51}$$

$$-c\, X + (s-d)\, Y = y_0 \tag{52}$$

and consequently

$$X = \frac{\det\begin{pmatrix} x_0 & -b \\ y_0 & s-d \end{pmatrix}}{\det\begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix}}; \qquad Y = \frac{\det\begin{pmatrix} x_0 & -b \\ y_0 & s-d \end{pmatrix}}{\det\begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix}}. \tag{53}$$

4. Unless you know what a "matrix exponential" is.

From this it is clear that there are three cases, depending on the form of the roots of $\det\begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix}=0$.

1. Two distinct real roots $r_1, r_2$. In this case the solution will involve $e^{r_1 t}$ and $e^{r_2 t}$.
2. One double root $r$. In this case the solution will involve $e^{rt}$ and $t\,e^{rt}$.
3. A pair of complex roots $\alpha \pm \beta i$. In this case the solution will involve $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$

We will not solve for the solutions here, since we will see that our understanding so far can lead to a simple way of solving such systems.

### 2.3. Two distinct real roots $r_1, r_2$.

In this case the solution took the form

$$x = \xi_1 e^{r_1 t} + \xi_2 e^{r_2 t}, \qquad y = \eta_1 e^{r_1 t} + \eta_2 e^{r_2 t} \tag{54}$$

or more compactly in the vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} e^{r_1 t} + \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} e^{r_2 t}. \tag{55}$$

Substituting into the equation, and obtain

$$\xi_1 r_1 e^{r_1 t} + \xi_2 r_2 e^{r_2 t} = a\,\xi_1 e^{r_1 t} + a\,\xi_2 e^{r_2 t} + b\,\eta_1 e^{r_1 t} + b\,\eta_2 e^{r_2 t} \tag{56}$$
$$\eta_1 r_1 e^{r_1 t} + \eta_2 r_2 e^{r_2 t} = c\,\xi_1 e^{r_1 t} + c\,\xi_2 e^{r_2 t} + d\,\eta_1 e^{r_1 t} + d\,\eta_2 e^{r_2 t} \tag{57}$$

which leads to

$$(r_1 - a)\,\xi_1 - b\,\eta_1 = 0 \tag{58}$$
$$-c\,\xi_1 + (r_1 - d)\,\eta_1 = 0 \tag{59}$$

and

$$(r_2 - a)\,\xi_2 - b\,\eta_2 = 0 \tag{60}$$
$$-c\,\xi_2 + (r_2 - d)\,\eta_2 = 0 \tag{61}$$

Naïvely one may conclude that $\xi_1 = \eta_1 = 1$, but remember that $r_1$ is such that $\det\begin{pmatrix} r_1-a & -b \\ -c & r_1-d \end{pmatrix}=0$, which makes the two equations into one – as long as $\xi_1, \eta_1$ satisfy one equation, they also satisfy the other.

**Remark 7.** Those who know linear algebra should have already recognized that $r_1$ is an "eigenvalue" of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}$ is the corresponding "eigenvector".

**Remark 8.** Note that two arbitrary constants are involved. See example below.

**Example 9.** Solve

$$\dot{x} = 3\,x - 2\,y \tag{62}$$
$$\dot{y} = 2\,x - 2\,y. \tag{63}$$

**Solution.** First compute

$$\det\begin{pmatrix} r-3 & 2 \\ -2 & r+2 \end{pmatrix} = r^2 - r - 2 \Longrightarrow r_{1,2} = 2, -1. \tag{64}$$

Thus we know that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} e^{2t} + \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} e^{-t}. \tag{65}$$

- Compute $\xi_1, \eta_1$. We need to solve

$$(2-3)\,\xi_1 + 2\,\eta_1 = 0 \tag{66}$$
$$-2\,\xi_1 + (2+2)\,\eta_1 = 0 \tag{67}$$

which gives

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{68}$$

- Compute $\xi_2, \eta_2$. We need to solve

$$\begin{aligned} (-1-3)\,\xi_2 + 2\,\eta_2 &= 0 \quad (69)\\ -2\,\xi_2 + (-1+2)\,\eta_2 &= 0 \quad (70) \end{aligned}$$

which leads to

$$\begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{71}$$

So the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}. \tag{72}$$

**2.4. More complicated cases.**

**Repeated roots $r_1 = r_2 = r$.**

In this case

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} e^{rt} + \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} t\, e^{rt}. \tag{73}$$

Substituting into the equations, we reach

$$\begin{aligned} r\,\xi_1\, e^{rt} + \xi_2\, e^{rt} + r\,\xi_2\, t\, e^{rt} &= a\,\xi_1\, e^{rt} + a\,\xi_2\, t\, e^{rt} + b\,\eta_1\, e^{rt} + b\,\eta_2\, t\, e^{rt} \quad (74)\\ r\,\eta_1\, e^{rt} + \eta_2\, e^{rt} + r\,\eta_2\, t\, e^{rt} &= c\,\xi_1\, e^{rt} + c\,\xi_2\, t\, e^{rt} + d\,\eta_1\, e^{rt} + d\,\eta_2\, t\, e^{rt} \quad (75) \end{aligned}$$

As $e^{rt}$ and $t\,e^{rt}$ are linearly independent, we must have

$$\begin{aligned} r\,\xi_1 + \xi_2 &= a\,\xi_1 + b\,\eta_1 \quad (76)\\ r\,\eta_1 + \eta_2 &= c\,\xi_1 + d\,\eta_1 \quad (77) \end{aligned}$$

and

$$\begin{aligned} r\,\xi_2 &= a\,\xi_2 + b\,\eta_2 \quad (78)\\ r\,\eta_2 &= c\,\xi_2 + d\,\eta_2. \quad (79) \end{aligned}$$

Therefore $\xi_{1,2}$ and $\eta_{1,2}$ are determined as follows.

- First determine $\xi_2, \eta_2$:

$$\begin{aligned} (r-a)\,\xi_2 - b\,\eta_2 &= 0 \quad (80)\\ -c\,\xi_2 + (r-d)\,\eta_2 &= 0. \quad (81) \end{aligned}$$

- Then determine $\xi_1, \eta_1$:

$$\begin{aligned} (r-a)\,\xi_1 - b\,\eta_1 &= -\xi_2 \quad (82)\\ -c\,\xi_1 + (r-d)\,\eta_1 &= -\eta_2. \quad (83) \end{aligned}$$

**Example 10.** Solve

$$\begin{aligned} \dot{x} &= 3\,x - 4\,y \quad (84)\\ \dot{y} &= x - y. \quad (85) \end{aligned}$$

**Solution.** First compute

$$\det \begin{pmatrix} r-3 & 4 \\ -1 & r+1 \end{pmatrix} = r^2 - 2\,r + 1 \Longrightarrow r_{1,2} = 1. \tag{86}$$

The solution is therefore

$$\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} \xi_1 \\ \eta_1 \end{array}\right) e^t + \left(\begin{array}{c} \xi_2 \\ \eta_2 \end{array}\right) t\, e^t. \tag{87}$$

- Solve $\xi_2, \eta_2$.

$$\begin{aligned} (1-3)\,\xi_2 + 4\,\eta_2 &= 0 \quad (88)\\ -\,\xi_2 + (1+1)\,\eta_2 &= 0. \quad (89) \end{aligned}$$

  This gives

$$\left(\begin{array}{c} \xi_2 \\ \eta_2 \end{array}\right) = C_2 \left(\begin{array}{c} 2 \\ 1 \end{array}\right). \tag{90}$$

- Solve $\xi_1, \eta_1$.

$$\begin{aligned} -\,2\,\xi_1 + 4\,\eta_1 = (1-3)\,\xi_1 + 4\,\eta_1 &= -\,2\,C_2 \quad (91)\\ -\,\xi_1 + 2\,\eta_1 = -\,\xi_1 + (1+1)\,\eta_1 &= -\,C_2. \quad (92) \end{aligned}$$

  This leads to

$$\xi_1 = 2\,\eta_1 + C_2 \tag{93}$$

  which gives

$$\left(\begin{array}{c} \xi_1 \\ \eta_1 \end{array}\right) = \left(\begin{array}{c} 2\,C_1 + C_2 \\ C_1 \end{array}\right). \tag{94}$$

Putting things together, we have

$$\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} 2\,C_1 + C_2 \\ C_1 \end{array}\right) e^t + C_2 \left(\begin{array}{c} 2 \\ 1 \end{array}\right) t\, e^t = C_1 \left(\begin{array}{c} 2 \\ 1 \end{array}\right) e^t + C_2 \left[\left(\begin{array}{c} 2 \\ 1 \end{array}\right) t\, e^t + \left(\begin{array}{c} 1 \\ 0 \end{array}\right) e^t\right]. \tag{95}$$

**A "degenerate" case (Thanks to Mr. Dahua Zeng for pointing this out).**

In this "double root" case, sometimes weird things happen.

**Example 11.** Solve

$$\begin{aligned} \dot{x} &= 3\,x \quad (96)\\ \dot{y} &= 3\,y \quad (97) \end{aligned}$$

**Solution.** The system is de-coupled so it is easy to get

$$\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} C_1\, e^{3t} \\ C_2\, e^{3t} \end{array}\right). \tag{98}$$

But we pretend we cannot do this and proceed in the "standard" way (it is now clear that something is different – there is no $t\, e^{3t}$ term!). We compute

$$\det\left(\begin{array}{cc} s-3 & 0 \\ 0 & s-3 \end{array}\right) = (s-3)^2 \Longrightarrow r = 3 \ \text{ double root.} \tag{99}$$

Write

$$\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} \xi_1 \\ \eta_1 \end{array}\right) e^{3t} + \left(\begin{array}{c} \xi_2 \\ \eta_2 \end{array}\right) t\, e^{3t}. \tag{100}$$

First solve $\xi_2, \eta_2$:

$$(3-3)\,\xi_2 - 0\,\eta_2 = 0; \qquad 0\,\xi_2 + (3-3)\,\eta_2 = 0. \tag{101}$$

We see that any $\xi_2, \eta_2$ are solutions!

On the other hand, when we try to compute $\xi_1, \eta_1$, we face

$$\begin{aligned} 0\,\xi_1 + 0\,\eta_1 &= -\,\xi_2 \quad (102)\\ 0\,\xi_1 + 0\,\eta_1 &= -\,\eta_2 \quad (103) \end{aligned}$$

It is clear that no solutions exist unless $\xi_2 = \eta_2 = 0$!

**Remark 12.** In our $2\times 2$ case, the above is actually the only situation where such things can happen[5], so it doesn't cause any practical problem, as the system is de-coupled and $x$, $y$ can be solved separately. However, for the general $n\times n$ case, things are not so simple. As a consequence, attempting to give general formulas in the $n\times n$ case without using the language of matrices is a pure waste of time.
Everything will become crystal clear as soon as one is familiar with matrices and in particular the so-called "Jordan canonical form".

**Complex roots $\alpha\pm\beta i$.**

In this case we have

$$\left(\begin{array}{c} x\\ y \end{array}\right)=\left(\begin{array}{c} \xi_1\\ \eta_1 \end{array}\right)e^{\alpha t}\cos\beta t+\left(\begin{array}{c} \xi_2\\ \eta_2 \end{array}\right)e^{\alpha t}\sin\beta t. \tag{110}$$

But the formulas get complicated. A better approach is to write

$$\left(\begin{array}{c} x\\ y \end{array}\right)=\left(\begin{array}{c} \xi_1\\ \eta_1 \end{array}\right)e^{r_1 t}+\left(\begin{array}{c} \xi_2\\ \eta_2 \end{array}\right)e^{r_2 t} \tag{111}$$

with $r_{1,2}=\alpha\pm i\,\beta$, following the formulas in the distinct roots case, and finally obtain real solutions through separating real and imaginary parts.

**Example 13.** Solve

$$\begin{aligned} \dot{x} &= 3\,x-2\,y && (112)\\ \dot{y} &= 4\,x-y. && (113) \end{aligned}$$

**Solution.** First solve the equation

$$\det\left(\begin{array}{cc} r-3 & 2\\ -4 & r+1 \end{array}\right)=0 \tag{114}$$

which is just

$$r^2-2\,r+5=0\Longrightarrow r_{1,2}=1\pm 2\,i. \tag{115}$$

Now we try to write the solution in the form

$$\left(\begin{array}{c} \xi_1\\ \eta_1 \end{array}\right)e^{(1+2i)t}+\left(\begin{array}{c} \xi_2\\ \eta_2 \end{array}\right)e^{(1-2i)t}. \tag{116}$$

- Solve $\xi_1,\eta_1$. They are solutions of

  $$\begin{aligned} [(1+2\,i)-3]\,\xi_1+2\,\eta_1 &= 0 && (117)\\ -4\,\xi_1+[(1+2\,i)+1]\,\eta_1 &= 0 && (118) \end{aligned}$$

  which simplifies to

  $$\begin{aligned} -(1-i)\,\xi_1+\eta_1 &= 0 && (119)\\ -2\,\xi_1+(1+i)\,\eta_1 &= 0. && (120) \end{aligned}$$

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5. More specifically, if a $2\times 2$ system is such that the standard procedure does not work, then it looks like

$$\begin{aligned} \dot{x} &= a\,x && (104)\\ \dot{y} &= a\,y. && (105) \end{aligned}$$

To see this, just notice that for a general system

$$\begin{aligned} \dot{x} &= a\,x+b\,y && (106)\\ \dot{y} &= c\,x+d\,y && (107) \end{aligned}$$

If $\det\left(\begin{smallmatrix} s-a & -b\\ -c & s-d \end{smallmatrix}\right)$ has repeated roots and the $\xi_2,\eta_2$ equation

$$\begin{aligned} (s-a)\,\xi_2-b\,\eta_2 &= 0 && (108)\\ -c\,\xi_2+(s-d)\,\eta_2 &= 0 && (109) \end{aligned}$$

is satisfied by any $\xi_2,\eta_2$, then the coefficients $s-a,-b,-c,s-d$ must all be 0 and $a=d$.

Using the first equation we get $\eta_1 = (1-i)\,\xi_1$. Therefore

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \tag{121}$$

where $C$ is an arbitrary constant.

- Solve $\xi_2, \eta_2$. They are solutions of

$$[(1-2\,i)-3]\,\xi_2 + 2\,\eta_2 = 0 \tag{122}$$

$$-\,4\,\xi_2 + [(1-2\,i)+1]\,\eta_2 = 0 \tag{123}$$

which simplifies to

$$-\,(1+i)\,\xi_2 + \eta_2 = 0 \tag{124}$$

$$-\,2\,\xi_2 + (1-i)\,\eta_2 = 0 \tag{125}$$

yielding

$$\begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = C_2 \begin{pmatrix} 1 \\ 1+i \end{pmatrix}. \tag{126}$$

Now the solution is given by

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= C_1 \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t} + C_2 \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1-2i)t} \\ &= e^t \begin{pmatrix} C_1\,(\cos 2\,t + i \sin 2\,t) + C_2\,(\cos 2\,t - i \sin 2\,t) \\ C_1\,(1-i)\,(\cos 2\,t + i \sin 2\,t) + C_2\,(1+i)\,(\cos 2\,t - i \sin 2\,t) \end{pmatrix} \\ &= e^t \begin{pmatrix} (C_1 + C_2)\cos 2\,t + i\,(C_1 - C_2)\sin 2\,t \\ (C_1 + C_2)\,(\cos 2\,t + \sin 2\,t) + i\,(C_2 - C_1)\,(\cos 2\,t - \sin 2\,t) \end{pmatrix} \\ &= e^t\,(C_1 + C_2) \begin{pmatrix} \cos 2\,t \\ \cos 2\,t + \sin 2\,t \end{pmatrix} + i\,e^t\,(C_1 - C_2) \begin{pmatrix} \sin 2\,t \\ \sin 2\,t - \cos 2\,t \end{pmatrix}. \end{aligned} \tag{127}$$

As $C_1, C_2$ are arbitrary constants, $C_1' = C_1 + C_2$, $C_2' = C_1 - C_2$ are also arbitrary constants. Therefore the real solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1'\,e^t \begin{pmatrix} \cos 2\,t \\ \cos 2\,t + \sin 2\,t \end{pmatrix} + C_2'\,e^t \begin{pmatrix} \sin 2\,t \\ \sin 2\,t - \cos 2\,t \end{pmatrix}. \tag{128}$$