MATH 334 A1 HOMEWORK 4 (DUE NOV. 26 5PM)

- No "Advanced" or "Challenge" problems will appear in homeworks.

BASIC PROBLEMS

Problem 1. (6.1.9) Find the Laplace transform of

$$f(t) = e^{a t} \cosh b\, t \tag{1}$$

where $\cosh b\, t$ is defiend as $\left(e^{b t} + e^{-b t}\right)/2$.
Solution. We have

$$f(t) = e^{a t} \left(\frac{e^{b t} + e^{-b t}}{2}\right) = \frac{1}{2} e^{(a+b) t} + \frac{1}{2} e^{(a-b) t}. \tag{2}$$

Therefore

$$\mathcal{L}(f)(s) = \mathcal{L}\left\{\frac{1}{2} e^{(a+b) t} + \frac{1}{2} e^{(a-b) t}\right\} = \frac{1}{2} \mathcal{L}\left\{e^{(a+b) t}\right\} + \frac{1}{2} \mathcal{L}\left\{e^{(a-b) t}\right\} = \frac{1}{2}\left[\frac{1}{s-a-b} + \frac{1}{s-a+b}\right] = \frac{s-a}{(s-a)^2 - b^2}. \tag{3}$$

The domain is such that $s > a + b$ and $s > a - b$ both hold, which can be written as $s > a + |b|$.

Problem 2. (6.2 1) Find the inverse Laplace transform of

$$F(s) = \frac{3}{s^2 + 4}. \tag{4}$$

Solution. Comparing with the transformation table we realize that

$$\mathcal{L}^{-1}\{F\} = \frac{3}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 2^2}\right\} = \frac{3}{2} \sin 2\, t.$$

Problem 3. (6.2 5) Find the inverse Laplace transform of

$$F(s) = \frac{2\, s + 2}{s^2 + 2\, s + 5}. \tag{5}$$

Solution. Comparing with the transformation table we realize that

$$\mathcal{L}^{-1}\{F\} = 2\, \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 2^2}\right\} = 2\, e^{-t} \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} = 2\, e^{-t} \cos 2\, t. \tag{6}$$

Problem 4. (6.2 8) Find the inverse Laplace transform of

$$F(s) = \frac{8\, s^2 - 4\, s + 12}{s\, (s^2 + 4)}. \tag{7}$$

Solution. We use partial fractions:

$$F(s) = \frac{A}{s} + \frac{B\, s + C}{s^2 + 4} = \frac{A\left(s^2 + 4\right) + s\, (B\, s + C)}{s\, (s^2 + 4)} = \frac{(A + B)\, s^2 + C\, s + 4\, A}{s\, (s^2 + 4)}. \tag{8}$$

Thus A, B, C are determined through

$$A + B = 8; \qquad C = -4; \qquad 4\, A = 12 \tag{9}$$

which gives

$$A = 3, \quad B = 5, \quad C = -4. \tag{10}$$

So

$$\begin{aligned}
\mathcal{L}^{-1}\{F\} &= \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5\, s - 4}{s^2 + 4}\right\} && (11)\\
&= \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} + 5\, \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} - 2\, \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 2^2}\right\} && (12)\\
&= 3 + 5 \cos 2\, t - 2 \sin 2\, t. && (13)
\end{aligned}$$

Problem 5. (6.2 12) Use Laplace transform to solve

$$y'' + 3\, y' + 2\, y = 0; \qquad y(0) = 1, \quad y'(0) = 0. \tag{14}$$

Solution. We follow the standard procedure:

1. Transform the equation:

$$\mathcal{L}\{y'' + 3\, y' + 2\, y\} = \mathcal{L}\{0\} \tag{15}$$

The left hand side is (we will use Y to denote $\mathcal{L}\{y\}$)

$$\begin{aligned}\mathcal{L}\{y''+3\,y'+2\,y\} &= \mathcal{L}\{y''\}+3\,\mathcal{L}\{y'\}+2\,\mathcal{L}\{y\}\\ &= s^2\,Y-s\,y(0)-y'(0)+3\,s\,Y-3\,y(0)+2\,Y\\ &= \left(s^2+3\,s+2\right)Y-s-3.\end{aligned} \tag{16}$$

The transformed equation is

$$\left(s^2+3\,s+2\right)Y-s-3=0. \tag{17}$$

2. Solve Y.

$$Y=\frac{s+3}{s^2+3\,s+2}. \tag{18}$$

3. Take inverse transform. First factorize the denominator: $s^2+3\,s+2=(s+1)\,(s+2)$, no complex roots, no repeated root. Thus the partial fraction representation should be

$$\frac{s+3}{s^2+3\,s+2}=\frac{A}{s+1}+\frac{B}{s+2}=\frac{(A+B)\,s+B+2\,A}{(s+1)\,(s+2)}. \tag{19}$$

This gives

$$A+B=1, \qquad B+2\,A=3\Longrightarrow A=2, B=-1. \tag{20}$$

So

$$\frac{s+3}{s^2+3\,s+2}=\frac{2}{s+1}-\frac{1}{s+2}. \tag{21}$$

Taking the inverse transform, we reach

$$y=\mathcal{L}^{-1}\{Y\}=2\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}=2\,e^{-t}-e^{-2t}. \tag{22}$$

Problem 6. (6.2 17) Use Laplace transform to solve

$$y^{(4)}-4\,y'''+6\,y''-4\,y'+y=0; \qquad y(0)=0, \quad y'(0)=1, \quad y''(0)=0, \quad y'''(0)=1. \tag{23}$$

Solution.

1. Transform the equation:

$$\begin{aligned}\mathcal{L}\left\{y^{(4)}-4\,y'''+6\,y''-4\,y'+y\right\} &= s^4\,Y-s^3\,y(0)-s^2\,y'(0)-s\,y''(0)-y'''(0)\\ &\quad -4\left[s^3\,Y-s^2\,y(0)-s\,y'(0)-y''(0)\right]\\ &\quad +6\left[s^2\,Y-s\,y(0)-y'(0)\right]-4\,[s\,Y-y(0)]+Y\\ &= \left(s^4-4\,s^3+6\,s^2-4\,s+1\right)Y-s^2-1-4\,[-s]-6\\ &= \left(s^4-4\,s^3+6\,s^2-4\,s+1\right)Y-s^2+4\,s-7.\end{aligned} \tag{24}$$

So the transformed equation is

$$\left(s^4-4\,s^3+6\,s^2-4\,s+1\right)Y-s^2+4\,s-7=0. \tag{25}$$

2. Solve Y.

$$Y=\frac{s^2-4\,s+7}{s^4-4\,s^3+6\,s^2-4\,s+1}. \tag{26}$$

3. Compute the inverse transform. To do this we need to first factorize the denominator. It is easy to see that 1 is a root, thus we write

$$s^4-4\,s^3+6\,s^2-4\,s+1=(s-1)\left(s^3-3\,s^2+3\,s-1\right). \tag{27}$$

Then clearly $s=1$ is also a root of $s^3-3\,s^2+3\,s-1$, so we have

$$s^3-3\,s^2+3\,s-1=(s-1)\left(s^2-2\,s+1\right)=(s-1)^3. \tag{28}$$

Thus

$$s^4-4\,s^3+6\,s^2-4\,s+1=(s-1)^4. \tag{29}$$

We need to find A, B, C, D such that

$$\frac{s^2-4\,s+7}{s^4-4\,s^3+6\,s^2-4\,s+1}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}+\frac{D}{(s-1)^4}=\frac{A\,(s-1)^3+B\,(s-1)^2+C\,(s-1)+D}{(s-1)^4}. \tag{30}$$

For this problem, it is easy to see that

$$s^2-4\,s+7=(s-1)^2-2\,(s-1)+4. \tag{31}$$

Therefore

$$A=0, B=1, C=-2, D=4. \tag{32}$$

We compute

$$\begin{aligned} y = \mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}\right\} \\ &= e^t\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}\right] \\ &= e^t\left[t - t^2 + \frac{2}{3}t^3\right]. \end{aligned} \tag{33}$$

Intermediate Problems

Problem 7. (5.5 8) Consider

$$2\,x^2\,y'' + 3\,x\,y' + \left(2\,x^2 - 1\right)\,y = 0. \tag{34}$$

a) Show that the equation has a regular singular point at $x = 0$;

b) Determine the indicial equation, the recurrence relation, and the roots of the indicial equation;

c) Find the series solution ($x > 0$) corresponding to the larger root;

d) If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

Solution.

a) Write the equation into standard form

$$y'' + \frac{3}{2}\frac{1}{x}\,y' + \frac{2\,x^2 - 1}{2\,x^2}\,y = 0. \tag{35}$$

It is clear that $x = 0$ is a singular point. To check whether it is regular singular, we compute

$$x\,p = \frac{3}{2}; \qquad x^2\,q = \frac{2\,x^2 - 1}{2}. \tag{36}$$

Both are analytic at 0 as both are polynomials. Therefore $x = 0$ is a regular singular point.

b) To determine the indicial equation, we need to find out p_0, q_0 which are the constant terms in the expansion of $x\,p$ and $x^2\,q$. A simple way to do this is set $x = 0$:

$$p_0 = (x\,p)\,|_{x=0} = \frac{3}{2}, \qquad q_0 = \left(x^2\,q\right)|_{x=0} = -\frac{1}{2}. \tag{37}$$

The indicial equation is

$$r\,(r-1) + p_0\,r + q_0 = r^2 + \frac{1}{2}r - \frac{1}{2} = 0 \Longrightarrow r_1 = \frac{1}{2}, r_2 = -1. \tag{38}$$

To find the recurrence relation we substitute (of course the following is not necessary for anyone who can remember the general formula for recurrence relations)

$$y = x^r \sum_{n=0}^{\infty} a_n\,x^n = \sum_{n=0}^{\infty} a_n\,x^{n+r} \tag{39}$$

into the equation. We get[1]

$$x^2 \sum_{n=0}^{\infty} 2\,(n+r)\,(n+r-1)\,a_n\,x^{n+r-2} + 3\,x \sum_{n=0}^{\infty} (n+r)\,a_n\,x^{n+r-1} + 2\,x^2 \sum_{n=0}^{\infty} a_n\,x^{n+r} - \sum_{n=0}^{\infty} a_n\,x^{n+r} = 0. \tag{40}$$

Simplify a bit, we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} 2\,(n+r)\,(n+r-1)\,a_n\,x^{n+r} + \sum_{n=0}^{\infty} 3\,(n+r)\,a_n\,x^{n+r} - \sum_{n=0}^{\infty} a_n\,x^{n+r} \\ &\quad + 2\sum_{n=0}^{\infty} a_n\,x^{n+r+2} \\ &= \sum_{n=0}^{\infty}\left[2\,(n+r)\left(n+r+\frac{1}{2}\right) - 1\right]a_n\,x^{n+r} + 2\sum_{n=2}^{\infty} a_{n-2}\,x^{n+r} \\ &= (2\,r\,(r+1/2) - 1)\,a_0 + [2\,(1+r)\,(1+r+1/2) - 1]\,a_1 \\ &\quad + \sum_{n=0}^{\infty}\left\{\left[2\,(n+r)\left(n+r+\frac{1}{2}\right) - 1\right]a_n + 2\,a_{n-2}\right\}x^{n+r}. \end{aligned} \tag{41}$$

1. For problems of the form $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$ with P, Q, R simple polynomials, usually it's simpler to substitute the expansion into this equation than into the one in standard form $y'' + (Q/P)\,y' + (R/P)\,y = 0$. Of course, no "substitution" is needed for those who can remember the general formula.

This gives

- Indicial equation ($n=0$):
$$2\,r\,(r+1/2)-1=0; \tag{42}$$
- $n=1$:
$$a_1=0. \tag{43}$$
- Recurrence relation for $n\geqslant 2$:
$$a_n=-\frac{2\,a_{n-2}}{2\,(n+r)\left(n+r+\frac{1}{2}\right)-1}. \tag{44}$$
Careful calculation gives
$$2\,(n+r)\,(n+r+1/2)-1=2\,(n+r)^2+(n+r)-1=(n+r+1)\,(2\,n+2\,r-1). \tag{45}$$
So a better formula for the recurrence relation is
$$a_n=-\frac{2}{(n+r+1)\,(2\,n+2\,r-1)}\,a_n. \tag{46}$$
- Note that following this recurrence relation, we have $a_1=a_3=a_5=\cdots=0$.

c) The larger root is $1/2$. Setting $r=1/2$ we have
$$a_n=-\frac{1}{(n+3/2)\,n}\,a_{n-2} \tag{47}$$
which leads to (setting $a_0=1$)
$$a_2=-\frac{1}{7},\quad a_4=\frac{1}{2!\,7\cdot 11},\qquad a_{2m}=\frac{(-1)^m}{m!\,7\cdot 11\cdot\cdots\cdot(4\,m+3)}; \tag{48}$$
The first solution is given by
$$y_1=x^{1/2}\left[1-\frac{x^2}{7}+\frac{x^4}{2!\,7\cdot 11}+\cdots+\frac{(-1)^m}{m!\,7\cdot 11\cdot\cdots\cdot(4\,m+3)}\,x^{2m}+\cdots\right]=x^{1/2}\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!\,7\cdot 11\cdot\cdots\cdot(4\,m+3)}\,x^{2m}\right].$$

d) As $1/2-(-1)=3/2$ is not an integer, setting $r=-1$ gives us the 2nd solution
$$y_2=x^{-1}\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m\,x^{2m}}{(m!)\,5\cdot 9\cdot\cdots\cdot(4\,m-3)}\right]. \tag{49}$$

Problem 8. (6.1 22) Determine whether
$$\int_0^{\infty} t\,e^{-t}\,\mathrm{d}t \tag{50}$$
converges or diverges.

Solution.

- Method 1. Integration by parts, we have
$$\int_0^{\infty} t\,e^{-t}\,\mathrm{d}t=-\int_0^{\infty} t\,\mathrm{d}e^{-t}=-t\,e^{-t}\,|_0^{\infty}+\int_0^{\infty} e^{-t}\,\mathrm{d}t=1 \tag{51}$$
so the integral converges.
- Method 2. Break the integral into
$$\int_0^{\infty} t\,e^{-t}\,\mathrm{d}t=\int_0^{T} t\,e^{-t}\,\mathrm{d}t+\int_T^{\infty} t\,e^{-t}\,\mathrm{d}t. \tag{52}$$
The first integral clearly converges no matter what T is. Now choose T large enough so that $t\,e^{-t}<e^{-t/2}$ for all $t>T$ we see that the second integral also converges.