Math 334 A1 Homework 3 (Due Nov. 5 5pm)

- No "Advanced" or "Challenge" problems will appear in homeworks.

Basic Problems

Problem 1. (4.1 11) Verify that the given functions are solutions of the differential equation, and determine their Wronskian.

$$y''' + y' = 0; \qquad 1,\ \cos t,\ \sin t. \tag{1}$$

Solution. We compute

$$(1)''' + (1)' = 0 + 0 = 0; \tag{2}$$

$$(\cos t)''' + (\sin t)' = -\cos t + \cos t = 0; \tag{3}$$

$$(\sin t)''' + (\sin t)' = -\cos t + \cos t = 0. \tag{4}$$

Compute the Wronskian:

$$W = \det \begin{pmatrix} 1 & \cos t & \sin t \\ (1)' & (\cos t)' & (\sin t)' \\ (1)'' & (\cos t)'' & (\sin t)'' \end{pmatrix} = \det \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix} = \sin^2 t + \cos^2 t = 1. \tag{5}$$

Problem 2. (4.2 1) Express $1 + i$ in the form $R\,(\cos\theta + i\sin\theta) = R\,e^{i\theta}$.
Solution. We need

$$R\cos\theta = 1, \qquad R\sin\theta = 1. \tag{6}$$

Therefore

$$R^2 = 2 \Longrightarrow R = \sqrt{2}. \tag{7}$$

This gives

$$\cos\theta = \frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}} \Longrightarrow \theta = \frac{\pi}{4} + 2\,k\,\pi \tag{8}$$

where k can be any integer.

Therefore

$$1 + i = R\left(\cos\left(\frac{\pi}{4} + 2\,k\,\pi\right) + i\sin\left(\frac{\pi}{4} + 2\,k\,\pi\right)\right) = R\,e^{i\left(\frac{\pi}{4} + 2k\pi\right)}. \tag{9}$$

Problem 3. (4.2 9) Find all four roots of $1^{1/4}$.
Solution. To find all roots, we need to write 1 into the form $R\,e^{i\theta}$. Clearly $R = 1$, $\cos\theta = 1$, $\sin\theta = 0$ thus

$$1 = e^{2k\pi i}, \qquad k \text{ is any integer.} \tag{10}$$

Now we have

$$1^{1/4} = e^{(2k\pi i)/4} = e^{\frac{k\pi}{2} i}. \tag{11}$$

It is clear that k and $k + 4$ gives the same root for any k. Therefore the four roots are given by $k = 0, 1, 2, 3$. Setting $k = 0$ we obtain 1; Setting $k = 1$ we obtain $e^{\frac{\pi}{2} i} = i$; Setting $k = 2$ we obtain -1; Setting $k = 3$ we obtain $-i$. So finally the four roots are

$$1, i, -1, -i. \tag{12}$$

Problem 4. (5.1 7) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\,n^2\,(x+2)^n}{3^n}. \tag{13}$$

Solution. We have

$$a_n = \frac{(-1)^n\,n^2}{3^n}. \tag{14}$$

Thus

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\frac{(n+1)^2}{n^2}. \tag{15}$$

Taking the limit $n \nearrow \infty$, we have

$$L = \lim_{n \longrightarrow \infty} \frac{1}{3}\frac{(n+1)^2}{n^2} = \frac{1}{3}. \tag{16}$$

Therefore the radius of convergence is

$$\rho = L^{-1} = 3. \tag{17}$$

Problem 5. (5.1 13) Determine the Taylor series about x_0 for the given function:

$$y(x) = \ln x, \qquad x_0 = 1. \tag{18}$$

Solution. Recall that the Taylor series is given by

$$y(x) = y(x_0) + y'(x_0)\,(x - x_0) + \frac{y''(x_0)}{2}\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{19}$$

Now $y = \ln x$ and $x_0 = 1$. We compute for $n \geqslant 1$

$$y^{(n)}(x_0) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\ln x)\,|_{x=x_0} = (-1)^{n+1}\,(n-1)!\,x^{-n}\,|_{x=x_0=1} = (-1)^{n+1}\,(n-1)!. \tag{20}$$

Note that $y(x_0) = \ln 1 = 0$.

So the desired Taylor series is

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,(x-1)^n. \tag{21}$$

Problem 6. (5.1 21) Rewrite the given expression as a sum whose generic term involves x^n:

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}. \tag{22}$$

Solution. WE need to shift $n - 2 \longrightarrow n$. This means the sum now starts from 0, and n becomes $n + 2$. So the sum becomes

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n. \tag{23}$$

Problem 7. (5.2 3) Consider

$$y'' - x\,y' - y = 0, \qquad x_0 = 1, \tag{24}$$

a) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

b) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions (that is y_1, y_2 are linearly independent.)

Solution.

a) Write

$$y = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n = \sum_{n=0}^{\infty} a_n\,(x-1)^n. \tag{25}$$

Substitute into the equation, we have

$$\left(\sum_{n=0}^{\infty} a_n\,(x-1)^n\right)'' - [(x-1)+1]\left(\sum_{n=0}^{\infty} a_n(x-1)^n\right)' - \sum_{n=0}^{\infty} a_n\,(x-1)^n = 0. \tag{26}$$

First compute the first term:

$$\left(\sum_{n=0}^{\infty} a_n\,(x-1)^n\right)'' = \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,(x-1)^{n-2}. \tag{27}$$

Shifting index, we reach

$$\left(\sum_{n=0}^{\infty} a_n\,(x-1)^n\right)'' = \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,(x-1)^n. \tag{28}$$

Now compute the second term

$$-[(x-1)+1]\left(\sum_{n=0}^{\infty} a_n(x-1)^n\right)' = -(x-1)\sum_{n=1}^{\infty} n\,a_n\,(x-1)^{n-1} - \sum_{n=1}^{\infty} n\,a_n\,(x-1)^{n-1} \tag{29}$$

$$= -\sum_{n=1}^{\infty} n\,a_n\,(x-1)^n - \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,(x-1)^n. \tag{30}$$

Now the equation becomes

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,(x-1)^n - \sum_{n=1}^{\infty} n\,a_n\,(x-1)^n - \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,(x-1)^n - \sum_{n=0}^{\infty} a_n\,(x-1)^n = 0. \tag{31}$$

Note that in the above, three sums start from 0 while one starts from 1. Thus we need to write the $n = 0$ term separately:

$$2\,a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} [(n+2)\,(n+1)\,a_{n+2} - n\,a_n - (n+1)\,a_{n+1} - a_n] = 0. \tag{32}$$

The recurrence relations are

$$2\,a_2 - a_1 - a_0 = 0 \tag{33}$$

$$(n+2)\,(n+1)\,a_{n+2} - (n+1)\,a_n - (n+1)\,a_{n+1} = 0 \qquad n \geqslant 1 \tag{34}$$

The second relation can be simplified to

$$(n+2)\,a_{n+2}=a_n+a_{n+1}. \qquad n\geqslant 1 \tag{35}$$

Solving them one by one, we have

$$(n=0) \qquad a_2 = \frac{1}{2}a_0+\frac{1}{2}a_1 \tag{36}$$

$$(n=1) \qquad a_3 = \frac{1}{3}(a_1+a_2)=\frac{1}{6}a_0+\frac{1}{2}a_1 \tag{37}$$

$$(n=2) \qquad a_4 = \frac{1}{4}(a_2+a_3)=\frac{1}{4}\left(\frac{2}{3}a_0+a_1\right)=\frac{1}{6}a_0+\frac{1}{4}a_1 \tag{38}$$

The general solution is

$$y(x)=a_0+a_1(x-1)+\left(\frac{1}{2}a_0+\frac{1}{2}a_1\right)(x-1)^2+\left(\frac{1}{6}a_0+\frac{1}{2}a_1\right)(x-1)^3+\left(\frac{1}{6}a_0+\frac{1}{4}a_1\right)(x-1)^4+\cdots \tag{39}$$

Collecting all the a_0's and the a_1's together we have

$$y(x) = a_0\left[1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots\right]+a_1\left[x-1+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4+\cdots\right]. \tag{40}$$

So

$$y_1(x) = 1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots \tag{41}$$

$$y_2(x) = x-1+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4+\cdots \tag{42}$$

b) The Wronskian at x_0 is

$$\det\begin{pmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{pmatrix}=\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=1\neq 0. \tag{43}$$

So y_1, y_2 are linearly independent.

Problem 8. (5.2 15) Find the first five nonzero terms in the solution of the problem

$$y''-x\,y'-y=0, \qquad y(0)=2, \quad y'(0)=1. \tag{44}$$

Solution. Write

$$y=\sum_{n=0}^{\infty}a_n\,x^n. \tag{45}$$

Substitute into the equation:

$$0 = \left(\sum_{n=0}^{\infty}a_n\,x^n\right)''-x\left(\sum_{n=0}^{\infty}a_n\,x^n\right)'-\left(\sum_{n=0}^{\infty}a_n\,x^n\right) \tag{46}$$

$$= \sum_{n=2}^{\infty}n\,(n-1)\,a_n\,x^{n-2}-x\sum_{n=1}^{\infty}n\,a_n\,x^{n-1}-\sum_{n=0}^{\infty}a_n\,x^n \tag{47}$$

$$= \sum_{n=0}^{\infty}(n+2)\,(n+1)\,a_{n+2}\,x^n-\sum_{n=1}^{\infty}n\,a_n\,x^n-\sum_{n=0}^{\infty}a_n\,x^n \tag{48}$$

$$= 2\,a_2-a_0+\sum_{n=1}^{\infty}\left[(n+2)\,(n+1)\,a_{n+2}-(n+1)\,a_n\right]x^n. \tag{49}$$

Thus the recurrence relations are

$$2\,a_2-a_0 = 0, \tag{50}$$

$$(n+2)\,a_{n+2}-a_n = 0. \tag{51}$$

Now the initial conditions give

$$y(0)=2\Longrightarrow a_0=2; \qquad y'(0)=1\Longrightarrow a_1=1. \tag{52}$$

We compute

$$(n=0) \qquad a_2 = \frac{a_0}{2}=1; \tag{53}$$

$$(n=1) \qquad a_3 = \frac{a_1}{3}=\frac{1}{3}; \tag{54}$$

$$(n=2) \qquad a_4 = \frac{a_2}{4}=\frac{1}{4}. \tag{55}$$

We already have five nonzero terms:

$$y(x)=2+x+x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots \tag{56}$$

Problem 9. (5.3 7) Determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the differential equation

$$(1+x^3)\, y'' + 4\, x\, y' + 4\, y = 0; \qquad x_0 = 0,\ x_0 = 2. \tag{57}$$

Solution. Write the equation into standard form

$$y'' + \frac{4\, x}{1+x^3}\, y' + \frac{4}{1+x^3}\, y = 0. \tag{58}$$

We see that the singular points are solutions to

$$x^3 + 1 = 0. \tag{59}$$

or equivalently

$$x^3 = -1. \tag{60}$$

To find all such x, we need to write $-1 = R\, e^{i\theta}$. Clearly $R = 1$. To determine θ we solve

$$\cos\theta = -1, \qquad \sin\theta = 0 \tag{61}$$

which gives $\theta = \pi + 2\, k\, \pi$. Thus the solutions are given by

$$x = e^{i\frac{2k+1}{3}\pi}. \tag{62}$$

Notice that k and $k+3$ gives the same x. Therefore the three roots are given by setting $k = 0, 1, 2$.

$$k = 0 \Longrightarrow x = e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}\, i; \;\; k = 1 \Longrightarrow x = -1; \;\; k = 2 \Longrightarrow x = \frac{1}{2} - \frac{\sqrt{3}}{2}\, i. \tag{63}$$

Now we discuss

- $x_0 = 0$. The distance from 4 to the three roots are:

$$\left| 0 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \right) \right| = 1 \tag{64}$$

$$|0 - (-1)| = 1; \tag{65}$$

$$\left| 0 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\, i \right) \right| = 1; \tag{66}$$

  The smallest distance is 1. So the radius of convergence is at least 1.

- $x_0 = 2$. The distances are

$$\left| 2 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \right) \right| = \left| \frac{3}{2} - \frac{\sqrt{3}}{2}\, i \right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}; \tag{67}$$

$$|2 - (-1)| = 3; \tag{68}$$

$$\left| 2 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\, i \right) \right| = \sqrt{3}. \tag{69}$$

  The smallest distance is $\sqrt{3}$. So the radius of convergence is $\sqrt{3}$.

Problem 10. (5.3 12) Find the first four nonzero terms in each of two power series solutions about the origin for

$$e^x\, y'' + x\, y = 0 \tag{70}$$

Determine the lower bound of radius of convergence.

Solution. We write

$$y = \sum_{n=0}^{\infty} a_n\, x^n \tag{71}$$

and expand

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{72}$$

Substituting into the equation we have

$$0 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} a_n\, x^n \right)'' + x \sum_{n=0}^{\infty} a_n\, x^n \tag{73}$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \right) \left(2\, a_2 + 6\, a_3\, x + 12\, a_4\, x^2 + 20\, a_5\, x^3 + \cdots \right) + a_0\, x + a_1\, x^2 + a_2\, x^3 + \cdots \tag{74}$$

Note that in the above, we expand everything up to x^3, hoping that the recurrence relations would give us the desired four non-zero terms in both y_1 and y_2. If it turns out that this is not the case, we need to expand to higher order.

To make the calculation simpler, we notice that finally the solution is written as

$$y = a_0\, y_1 + a_1\, y_2. \tag{75}$$

Thus y_1 is obtained by setting $a_0=1, a_1=0$ while y_2 is obtained by setting $a_0=0, a_1=1$.

- Finding y_1. Setting $a_0=1, a_1=0$ we have

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)(2\,a_2+6\,a_3\,x+12\,a_4\;x^2+20\,a_5\,x^3+\cdots)+x+a_2\,x^3+\cdots=0 \tag{76}$$

Carrying out the multiplication, we have

$$2\,a_2+(2\,a_2+6\,a_3+1)\,x+(a_2+6\,a_3+12\,a_4)\,x^2+\left(\frac{a_2}{3}+3\,a_3+12\,a_4+20\,a_5+a_2\right)x^3+\cdots=0. \tag{77}$$

Thus we have

$$2\,a_2 = 0, \tag{78}$$
$$2\,a_2+6\,a_3+1 = 0, \tag{79}$$
$$a_2+6\,a_3+12\,a_4 = 0, \tag{80}$$
$$\frac{a_2}{3}+3\,a_3+12\,a_4+20\,a_5+a_2 = 0. \tag{81}$$

These give

$$a_2=0, \quad a_3=-\frac{1}{6}, \quad a_4=\frac{1}{12}, \quad a_5=-\frac{1}{40}. \tag{82}$$

Thus

$$y_1(x)=1-\frac{1}{6}\,x^3+\frac{1}{12}\,x^4-\frac{1}{40}\,x^5+\cdots \tag{83}$$

We are lucky that we have exactly four nonzero terms.

- Finding y_2. Setting $a_0=1$, $a_1=1$ we have

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)(2\,a_2+6\,a_3\,x+12\,a_4\;x^2+20\,a_5\,x^3+\cdots)+x^2+a_2\,x^3+\cdots=0. \tag{84}$$

Carrying out the multiplication, we have

$$2\,a_2+(2\,a_2+6\,a_3)\,x+(a_2+6\,a_3+12\,a_4+1)\,x^2+\left(\frac{a_2}{3}+3\,a_3+12\,a_4+20\,a_5+a_2\right)x^3+\cdots=0. \tag{85}$$

The recurrence relations are

$$2\,a_2 = 0, \tag{86}$$
$$2\,a_2+6\,a_3 = 0, \tag{87}$$
$$a_2+6\,a_3+12\,a_4+1 = 0, \tag{88}$$
$$\frac{a_2}{3}+3\,a_3+12\,a_4+20\,a_5+a_2 = 0, \tag{89}$$

which give

$$a_2=0; \quad a_3=0; \quad a_4=-\frac{1}{12}; \quad a_5=\frac{1}{20}. \tag{90}$$

Thus

$$y_2=x-\frac{1}{12}\,x^4+\frac{1}{20}\,x^5+\cdots \tag{91}$$

We only have 3 nonzero terms!

- Finding the 4th term.
  To find the 4th term, we need to expand everything to higher power. Let's try expanding to x^4:

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\cdots\right)(2\,a_2+6\,a_3\,x+12\,a_4\;x^2+20\,a_5\,x^3+30\,a_6\,x^4\cdots)+x^2+a_2\,x^3+a_3\,x^4\cdots=0. \tag{92}$$

This gives a new recurrence relation via setting coefficients of x^4 to be 0:

$$\frac{a_2}{12}+a_3+6\,a_4+20\,a_5+30\,a_6+a_3=0. \tag{93}$$

We obtain

$$a_6=-\frac{1}{60}. \tag{94}$$

The updated y_2 is now

$$y_2(x)=x-\frac{1}{12}\,x^4+\frac{1}{20}\,x^5-\frac{1}{60}\,x^6+\cdots \tag{95}$$

Now we have 4 nonzero terms.

To determine the lower bound of the radius of convergence, we need to find all z such that $e^z=0$, as the standard form of our equation is

$$y''+\frac{x}{e^x}\,y=0. \tag{96}$$

Write $z=\alpha+i\,\beta$. We have

$$e^z=e^\alpha\,[\cos\beta+i\sin\beta]. \tag{97}$$

Thus

$$|e^z| = e^{\alpha} \neq 0 \tag{98}$$

for any real number α. Therefore e^z is never zero and the equation does not have any singular point. Consequently the radius of convergence is ∞.

Problem 11. (5.4 1) Find the general solution

$$x^2\, y'' + 4\, x\, y' + 2\, y = 0. \tag{99}$$

Solution. This is Euler equation. Set $y = x^r$ we reach

$$r\,(r-1) + 4\, r + 2 = 0 \Longrightarrow r_{1,2} = -2, -1. \tag{100}$$

So the general solution is

$$y = C_1 x^{-2} + C_2\, x^{-1}. \tag{101}$$

Problem 12. (5.4 19) Find all singular points of

$$x^2\,(1-x)\, y'' + (x-2)\, y' - 3\, x\, y = 0, \tag{102}$$

and determine whether each one is regular or irregular.
Solution. Write the equation into standard form:

$$y'' + \frac{x-2}{x^2\,(1-x)}\, y' - \frac{3}{x\,(1-x)}\, y = 0. \tag{103}$$

We see that there are two singular points $x = 0, x = 1$.

- At $x = 0$, we have

$$x\, p = \frac{x-2}{x\,(1-x)}, \quad x^2\, q = -\frac{3\, x}{1-x}. \tag{104}$$

  We see that $x\, p$ is not analytic (still has singularity at 0). So $x = 0$ is an irregular singuar point.

- At $x = 1$, we have

$$(x-1)\, p = \frac{x-2}{x^2}, \quad (x-1)^2\, q = \frac{3\,(1-x)}{x} \tag{105}$$

  both are analytic at $x = 1$. So $x = 1$ is a regular singular point.

Intermediate Problems

Problem 13. (4.1 8) Determine whether the given set of functions is linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

$$f_1(t) = 2\, t - 3, \qquad f_2(t) = 2\, t^2 + 1, \qquad f_3(t) = 3\, t^2 + t. \tag{106}$$

(Note: As f_1, f_2, f_3 are not solutions to some 3rd order equation, Wronskian $\neq 0$ implies linear independence, but Wronskian $= 0$ does not imply linear dependence. Finding a "linear relation" means finding constants C_1, C_2, C_3 such that

$$C_1\, f_1 + C_2\, f_2 + C_3\, f_3 = 0. \tag{107}$$

)
Solution. We compute the Wronskian – if it $\neq 0$, the functions are linearly independent; If it $= 0$, we have to use other methods to determine.

$$W = \det \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix} = \det \begin{pmatrix} 2\, t - 3 & 2\, t^2 + 1 & 3\, t^2 + t \\ 2 & 4\, t & 6\, t + 1 \\ 0 & 4 & 6 \end{pmatrix} = 0. \tag{108}$$

Unfortunately this does not guarantee linear dependence of f_1, f_2, f_3. However, this indicates that we should try to show linear dependence.

As an alternative method, we try to directly find C_1, C_2, C_3 such that

$$C_1\,(2\, t - 3) + C_2\left(2\, t^2 + 1\right) + C_3\left(3\, t^2 + t\right) = 0. \tag{109}$$

As the left hand side is a polynomial – special case of power series — the above is equivalent to that coefficients for 1, t, t^2 all vanish. We rewrite the above equation to

$$(-3\, C_1 + C_2) + (2\, C_1 + C_3)\, t + (2\, C_2 + 3\, C_3)\, t^2 = 0. \tag{110}$$

This gives

$$-3\, C_1 + C_2 = 0 \tag{111}$$

$$2\, C_1 + C_3 = 0 \tag{112}$$

$$2\, C_2 + 3\, C_3 = 0. \tag{113}$$

Solving this system, we have

$$C_2 = 3\,C_1, \quad C_3 = -2\,C_1, \quad C_1 \text{ arbitrary} \tag{114}$$

So f_1, f_2, f_3 are linearly dependent, a linear relation is given by

$$f_1 + 3\,f_2 - 2\,f_3 = 0. \tag{115}$$

Problem 14. (4.2 11) Find the general solution of

$$y''' - y'' - y' + y = 0. \tag{116}$$

Solution. This is linear equation with constant coefficients. The characteristic equation is

$$r^3 - r^2 - r + 1 = 0. \tag{117}$$

Clearly $r = 1$ is a solution. Write

$$r^3 - r^2 - r + 1 = (r-1)\left(r^2 - 1\right) = (r-1)^2\,(r+1). \tag{118}$$

Therefore

$$r_{1,2} = 1; \qquad r_3 = -1. \tag{119}$$

So the solution is given by

$$y = C_1\,e^t + C_2\,t\,e^t + C_3\,e^{-t}. \tag{120}$$

Problem 15. (4.2 16) Find the general solution of

$$y^{(4)} - 5\,y'' + 4\,y = 0. \tag{121}$$

Solution. The characteristic equation is

$$r^4 - 5\,r^2 + 4 = 0. \tag{122}$$

Notice that if we set $R = r^2$, we have

$$R^2 - 5\,R + 4 = 0 \Longrightarrow R = 4, 1. \tag{123}$$

So the four roots are

$$r_{1,2,3,4} = \pm 2, \pm 1. \tag{124}$$

They are all different, so the general solution is given by

$$y(x) = C_1\,e^{2t} + C_2\,e^{-2t} + C_3\,e^t + C_4\,e^{-t}. \tag{125}$$