Math 334 A1 Homework 2 (Due Oct. 8 5pm)

- No "Advanced" or "Challenge" problems will appear in homeworks.

Basic Problems

Problem 1. (1.1 1) Consider

$$y' = 3 - 2\,y. \tag{1}$$

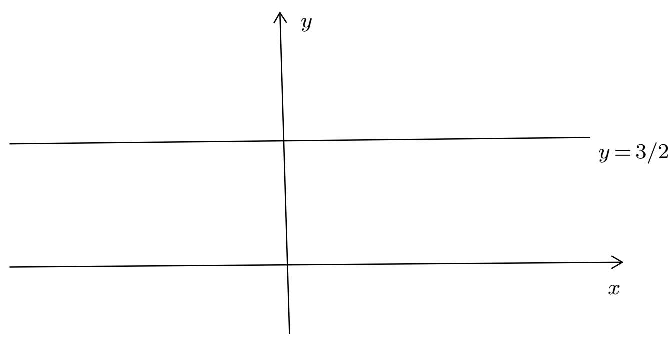
Determine the behavior of y as $t \longrightarrow \infty$ by drawing the direction field and analyze it.
Solution. First we find out points with $y' = 0$:

$$0 = y' = 3 - 2\,y \Longrightarrow y = 3/2. \tag{2}$$

Next we find out "straight-line" solutions, that is solutions of the form $y = a\,x + b$. Substituting into the equation we get

$$a = 3 - 2\,(a\,x + b) = 3 - 2\,b - 2\,a\,x. \tag{3}$$

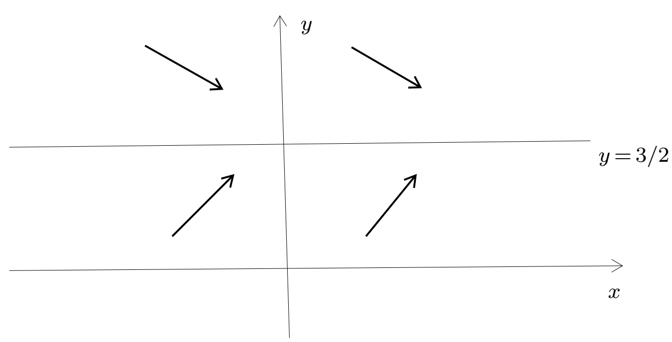
The only a, b such that this can hold are $a = 0$, $b = 3/2$.
Now we plot this constant solution $y = 3/2$:



Now the x-y plane is divided into two regions: $y > 3/2$ and $y < 3/2$. We know from uniqueness theorem that solutions can never cross each other, therefore any solution curve is either totally contained in $y > 3/2$ or totally contained in $y < 3/2$.

We also expect the behaviors of solutions in the same region are similar.

For the region $y > 3/2$, we have $y' = 3 - 2\,y < 0$. So the vectors look like $\searrow$ in this region; In contrast, they look like $\nearrow$ in the other region, $y < 3/2$. We add this information in:



Note that the exact length/direction of the vectors are not important, only the rough direction (in this problem, up or down) is important.

Looking at the vectors, we see that any solution in the upper region ($y > 3/2$) has to go down while any solution in the lower region has to go up. As they cannot cross the dividing line $y = 3/2$, any solution has to approach a limit.

The following is guaranteed by theory:

> Let $y = Y(x)$ be a solution of a first order ODE $y' = f(x, y)$. If $y \longrightarrow a$ as $x \longrightarrow \infty$, then $f(x, a) = 0$. In other words $y = a$ has to be a constant solution.

Now it's clear that the solutions of $y' = 3 - 2\,y$ all approach $3/2$ as $x \longrightarrow \infty$.

Problem 2. (1.3 27) Let

$$u_1(x, t) = \sin \lambda\,x \sin\lambda\,a\,t, \qquad u_2(x, t) = \sin(x - a\,t) \tag{4}$$

where λ is a real constant. Verify that u_1, u_2 are solutions of

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}. \tag{5}$$

Solution. We will only show that u_1 is a solution. The argument for u_2 is similar.

- u_1. We substitute u_1 into the left hand side:

$$a^2 \frac{\partial^2 u_1}{\partial x^2} = -a^2 \lambda^2 \sin \lambda x \sin \lambda a t; \tag{6}$$

Next substitute u_1 into the right hand side:

$$\frac{\partial^2 u_1}{\partial t^2} = -(a\lambda)^2 \sin \lambda x \sin \lambda a t. \tag{7}$$

Clearly

$$a^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}. \tag{8}$$

So u_1 is a solution.

Problem 3. (Ch.2 1) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 - 2y}{x}. \tag{9}$$

Solution. Notice that this equation is linear. We rewrite it to

$$y' + \frac{2}{x} y = x^2. \tag{10}$$

The integrating factor is then

$$e^{\int 2/x} = x^2. \tag{11}$$

We have

$$(x^2 y)' = x^2 y' + 2 x y = x^4. \tag{12}$$

Integrating we reach

$$x^2 y = \frac{1}{5} x^5 + C. \tag{13}$$

The solution is then given by

$$y = \frac{1}{5} x^3 + C x^{-2}. \tag{14}$$

(If one fails to recognize linearity, one has to rewrite it to

$$(x^3 - 2y)\,\mathrm{d}x - x\,\mathrm{d}y = 0 \tag{15}$$

find out that it's not exact, figure out the integrating factor, and then solve it.)

Problem 4. (Ch.2 3) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x + y}{3 + 3y^2 - x}, \qquad y(0) = 0. \tag{16}$$

Solution. This is an initial value problem. We need to first find the general solution, then use the initial value to determine the constant.

Clearly the equation is not linear. So we multiply both sides by $(3 + 3y^2 - x)\,\mathrm{d}x$ – The issue of extra solutions will be discussed later, to get

$$(2x + y)\,\mathrm{d}x - (3 + 3y^2 - x)\,\mathrm{d}y = 0. \tag{17}$$

As

$$\frac{\partial(2x + y)}{\partial y} = 1 = \frac{\partial[-(3 + 3y^2 - x)]}{\partial x}, \tag{18}$$

the equation is exact. We find $u(x, y)$ through

$$\frac{\partial u}{\partial x} = 2x + y \Longrightarrow u(x, y) = \int 2x + y\,\mathrm{d}x + g(y) = x^2 + xy + g(y). \tag{19}$$

$$-(3 + 3y^2 - x) = \frac{\partial u}{\partial y} = \frac{\partial(x^2 + xy + g(y))}{\partial y} = x + g'(y) \tag{20}$$

So

$$g'(y) = -3 - 3y^2 \Longrightarrow g(y) = -y^3 - 3y. \tag{21}$$

Thus

$$u(x, y) = x^2 + xy - y^3 - 3y \tag{22}$$

and the general solution to

$$(2x + y)\,\mathrm{d}x - (3 + 3y^2 - x)\,\mathrm{d}y = 0 \tag{23}$$

is

$$x^2+x\,y-y^3-3\,y=C. \tag{24}$$

Now we discuss the effect of multiplying $\left(3+3\,y^2-x\right)\mathrm{d}x$. (***To grader: Do not deduct any point if a student tried to discuss this issue but couldn't do it due to technical difficulties***) This may lead to extra solutions given by

$$3+3\,y^2-x=0 \Longrightarrow y=\pm\sqrt{\frac{x}{3}-1}. \tag{25}$$

This also gives

$$x\,y-3\,y=3\,y^3. \tag{26}$$

Combine with the general solution:

$$x^2+2\,y^3=C \tag{27}$$

to which clearly $y=\pm\sqrt{x/3-1}$ are not solutions. Therefore the multiplication does not introduce any extra solution. The general solution to the original problem is also

$$x^2+x\,y-y^3-3\,y=C. \tag{28}$$

Now we use the initial value to determine the constant. As $y(0)=0$, replacing x, y by 0 we reach

$$x^2+x\,y-y^3-3\,y=0. \tag{29}$$

Problem 5. (Ch.2 22) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2-1}{y^2+1}, \qquad y(-1)=1. \tag{30}$$

Solution. This equation is separable. Multiply both sides by $\left(y^2+1\right)\mathrm{d}x$ (note that this time no extra discussion is needed as $y^2+1\neq 0$):

$$\left(y^2+1\right)\mathrm{d}y=\left(x^2-1\right)\mathrm{d}x. \tag{31}$$

Integrating:

$$\frac{y^3}{3}+y=\frac{x^3}{3}-x+C. \tag{32}$$

Using the initial value:

$$\frac{1}{3}+1=-\frac{1}{3}+1+C \Longrightarrow C=\frac{2}{3}. \tag{33}$$

The solution is given by

$$\frac{y^3}{3}+y=\frac{x^3}{3}-x+\frac{2}{3}. \tag{34}$$

Problem 6. (3.1 1) Find the general solution:

$$y''+2\,y'-3\,y=0. \tag{35}$$

Solution. This is 2nd order linear homogeneous constant coefficient. Substituting $y=e^{rt}$ we obtain the characteristic equation

$$r^2+2\,r-3=0 \Longrightarrow r_1=-3, r_2=1. \tag{36}$$

The general solution is

$$y=C_1\,e^{-3t}+C_2\,e^t. \tag{37}$$

Problem 7. (3.1 9) Solve

$$y''+y'-2\,y=0, \qquad y(0)=1, \quad y'(0)=1. \tag{38}$$

Solution. The characteristic equation is

$$r^2+r-2=0 \Longrightarrow r_1=-2, r_2=1. \tag{39}$$

The general solution is

$$y=C_1\,e^{-2t}+C_2\,e^t. \tag{40}$$

Now use the initial values to determine the constants:

$$y(0)=1 \Longrightarrow C_1+C_2=1; \tag{41}$$

$$y'(0)=1 \Longrightarrow -2\,C_1+C_2=1. \tag{42}$$

Solving, we have

$$C_2=1, C_1=0. \tag{43}$$

Thus the solution is given by

$$y=e^t. \tag{44}$$

Problem 8. (3.3 7) Find the general solution

$$y''-2\,y'+2\,y=0. \tag{45}$$

Solution. The characteristic equation is

$$r^2 - 2\,r + 2 = 0 \Longrightarrow r_{1,2} = 1 \pm i. \tag{46}$$

So the general solution is

$$y = C_1\,e^t \cos t + C_2\,e^t \sin t. \tag{47}$$

Problem 9. (3.4 11) Solve

$$9\,y'' - 12\,y' + 4\,y = 0, \qquad y(0) = 2, \quad y'(0) = -1. \tag{48}$$

Solution. The characteristic equation is

$$9\,r^2 - 12\,r + 4 = 0 \Longrightarrow r_1 = r_2 = \frac{2}{3}. \tag{49}$$

The general solution is given by

$$y = C_1\,e^{\frac{2}{3}t} + C_2\,t\,e^{\frac{2}{3}t}. \tag{50}$$

Now use the initial values:

$$y(0) = 2 \Longrightarrow C_1 = 2; \tag{51}$$

$$y'(0) = -1 \Longrightarrow \frac{2}{3}\,C_1 + C_2 = -1. \tag{52}$$

We have

$$C_1 = 2, C_2 = -\frac{7}{3}. \tag{53}$$

The solution is given by

$$y = 2\,e^{\frac{2}{3}t} - \frac{7}{3}\,t\,e^{\frac{2}{3}t}. \tag{54}$$

Intermediate Problems

Problem 10. (2.2 31) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + x\,y + y^2}{x^2}. \tag{55}$$

Solution. This is a homogeneous equation. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + x\,y + y^2}{x^2} = \frac{(x^2 + x\,y + y^2)/x^2}{x^2/x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2. \tag{56}$$

Use the new unknown $v = y/x$. We have

$$x\,v' + v = 1 + v + v^2 \Longrightarrow x\,v' = 1 + v^2 \Longrightarrow \frac{\mathrm{d}v}{1 + v^2} = \frac{\mathrm{d}x}{x}. \tag{57}$$

Integrating, we get the solution:

$$\arctan v = \ln|x| + C \tag{58}$$

or

$$v = \tan(\ln|x| + C). \tag{59}$$

As $y = x\,v$, we finally get

$$y = x\tan(\ln|x| + C). \tag{60}$$

Problem 11. (2.4 28) Solve

$$t^2\,y' + 2\,t\,y - y^3 = 0. \tag{61}$$

Solution. First rewrite

$$y' + \frac{2}{t}\,y - \frac{1}{t^2}\,y^3 = 0. \tag{62}$$

This is a Bernoulli equation. divide both sides by y^3 (We will discuss $y = 0$ at the end):

$$y^{-3}\,y' + \frac{2}{t}\,y^{-2} = \frac{1}{t^2} \Longrightarrow -\frac{1}{2}\left(y^{-2}\right)' + \frac{2}{t}\,y^{-2} = \frac{1}{t^2}. \tag{63}$$

Setting $v = y^{-2}$ (the purpose is to make the formulas less mess, and thus reducing the possibility of typos):

$$-\frac{1}{2}\,v' + \frac{2}{t}\,v = \frac{1}{t^2} \Longrightarrow v' - \frac{4}{t}\,v = -\frac{2}{t^2}. \tag{64}$$

The integrating factor is

$$e^{\int -4/t} = t^{-4}. \tag{65}$$

The equation becomes

$$\left(t^{-4}\,v\right)' = -\frac{2}{t^6}. \tag{66}$$

Integrating:

$$t^{-4}v = \frac{2}{5}t^{-5} + C \Longrightarrow v = \frac{2}{5}t^{-1} + Ct^4. \tag{67}$$

As $v = y^{-2}$, we have

$$y^2 = \left(\frac{2}{5}t^{-1} + Ct^4\right)^{-1}. \tag{68}$$

Finally, as we have divided both sides of the original equation by y^3, there is a possible loss of solution $y = 0$. Clearly $y = 0$ is a solution to the original equation, and is not included in the above formula. So the solutions to the original equation is

$$y = \pm\left(\frac{2}{5}t^{-1} + Ct^4\right)^{-1} \qquad \text{and} \qquad y = 0. \tag{69}$$

(Note that the textbook forgot the solution $y = 0$).

Problem 12. (3.1 17) Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$.
Solution. The characteristic equation with roots 2 and -3 are

$$(r-2)(r-(-3)) = 0 \Longrightarrow r^2 + r - 6 = 0. \tag{70}$$

So the desired equation is

$$y'' + y' - 6y = 0. \tag{71}$$