Math 334 A1 Homework 1 (Due Sep. 24 5pm)

Sep. 17, 2010

- No "Advanced" or "Challenge" problems will appear in homeworks.

Basic Problems

Problem 1. (2.1 13) Solve

$$y' - y = 2\,t\,e^{2t}, \qquad y(0) = 1. \tag{1}$$

Solution. This is a linear equation in the form

$$y' + P(x)\,y = Q(x) \tag{2}$$

with $P = -1$ and $Q = 2\,t\,e^{2t}$. We need to multiply both sides by $e^{\int P}$ and then integrate.

First compute:

$$P = -1 \Longrightarrow \int P = -t \Longrightarrow e^{\int P} = e^{-t}. \tag{3}$$

Then check

$$e^{-t}\,y' - e^{-t}\,y = \left(e^{-t}\,y\right)'. \tag{4}$$

Now we need to integrate

$$\left(e^{-t}\,y\right)' = e^{-t}\,2\,t\,e^{2t} = 2\,t\,e^{t} \Longrightarrow e^{-t}\,y = \int\ 2\,t\,e^{t} + C. \tag{5}$$

To evaluate the integral $\int\ 2\,t\,e^{t}$, we need the "integration by parts" formula:

$$\int f\,\mathrm{d}g = f\,g - \int g\,\mathrm{d}f \tag{6}$$

with f, g functions. Thus we need to find f, g such that

$$\int\ 2\,t\,e^{t} = \int f\,\mathrm{d}g. \tag{7}$$

Recall that $e^{t} = \mathrm{d}e^{t}$, we try $f = 2\,t$, $g = e^{t}$.[1] We have

$$\int\ 2\,t\,\mathrm{d}e^{t} = 2\,t\,e^{t} - \int e^{t}\,\mathrm{d}(2\,t) = 2\,t\,e^{t} - 2\int e^{t}\,\mathrm{d}t = 2\,(t-1)\,e^{t}. \tag{8}$$

Thus

$$e^{-t}\,y = 2\,(t-1)\,e^{t} + C \Longrightarrow y = 2\,(t-1)\,e^{2t} + C\,e^{t}. \tag{9}$$

Check

$$y' - y = \left(2\,(t-1)\,e^{2t} + C\,e^{t}\right)' - \left(2\,(t-1)\,e^{2t} + C\,e^{t}\right) = 2\,e^{2t} + 4\,(t-1)\,e^{2t} - 2\,(t-1)\,e^{2t} = 2\,t\,e^{2t}. \tag{10}$$

So our general solution is correct.

Finally use the initial values to determine the constant C:

$$y(0) = 1 \Longrightarrow 1 = y(0) = 2\,(0-1)\,e^{2\cdot 0} + C\,e^{0} = C - 2 \Longrightarrow C = 3. \tag{11}$$

Therefore the solution is

$$y(t) = 2\,(t-1)\,e^{2t} + 3\,e^{t}.$$

Problem 2. (2.1 15) Solve

$$t\,y' + 2\,y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \qquad t > 0. \tag{12}$$

Solution. This is a linear equation. To solve it first we need to write it into the form

$$y' + P y = Q. \tag{13}$$

through dividing both sides by t:

$$y' + \frac{2}{t}\,y = t - 1 + \frac{1}{t}. \tag{14}$$

Now the integrating factor is

$$e^{\int P} = e^{\int\ 2/t} = e^{2\ln t} = e^{\ln t^2} = t^2. \tag{15}$$

We check

$$t^2\left(y' + \frac{2}{t}\,y\right) = \left(t^2\,y\right)' \tag{16}$$

1. Rule of thumb: Whenever $e^{a\,t}$ is involved and you plan to use integration by parts, try $g = e^{a\,t}$.

so the integrating factor is correct.

Now multiply the equation by t^2:

$$(t^2\, y)' = t^3 - t^2 + t. \tag{17}$$

Integrate:

$$t^2\, y = \frac{1}{4}\, t^4 - \frac{1}{3}\, t^3 + \frac{1}{2}\, t^2 + C \Longrightarrow y = \frac{1}{4}\, t^2 - \frac{1}{3}\, t + \frac{1}{2} + \frac{C}{t^2}. \tag{18}$$

Check that it is indeed correct:

$$t\, y' + 2\, y = t \left(\frac{1}{4}\, t^2 - \frac{1}{3}\, t + \frac{1}{2} + \frac{C}{t^2} \right)' + 2 \left(\frac{1}{4}\, t^2 - \frac{1}{3}\, t + \frac{1}{2} + \frac{C}{t^2} \right) = t^2 - t + 1. \tag{19}$$

Finally determine C using the initial value:

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \Longrightarrow C = \frac{1}{12}. \tag{20}$$

The solution is given by

$$y(t) = \frac{1}{4}\, t^2 - \frac{1}{3}\, t + \frac{1}{2} + \frac{1}{12\, t^2}. \tag{21}$$

Problem 3. (2.2 5) Solve

$$y' = (\cos^2 x)\, (\cos^2 2\, y). \tag{22}$$

Solution. This equation is separable:

$$y' = g(x)\, p(y) \tag{23}$$

with $g(x) = \cos^2 x,\ p(y) = \cos^2 2\, y$.

We divide both sides by $p(y) = \cos^2 2\, y$:

$$\frac{y'}{\cos^2 2\, y} = \cos^2 x. \tag{24}$$

Therefore

$$\int \frac{1}{\cos^2 2\, y}\, \mathrm{d}y = \int \cos^2 x\, \mathrm{d}x + C. \tag{25}$$

We evaluate the two integrals.

- $\int \frac{1}{\cos^2 2\, y}\, \mathrm{d}y$. Recall

  $$(\tan x)' = \frac{1}{\cos^2 x} \Longrightarrow \mathrm{d}(\tan y) = \frac{1}{\cos^2 y}\, \mathrm{d}y. \tag{26}$$

  To accomodate the $2\, y$ we try

  $$\mathrm{d}(\tan 2\, y) = \frac{1}{\cos^2 2\, y}\, \mathrm{d}(2\, y) = \frac{2}{\cos^2 2\, y}\, \mathrm{d}y. \tag{27}$$

  Thus

  $$\int \frac{1}{\cos^2 2\, y}\, \mathrm{d}y = \frac{1}{2} \tan 2\, y. \tag{28}$$

- $\int \cos^2 x\, \mathrm{d}x$. The standard methods is transforming $\cos^2 x$ to $\cos 2\, x$ using the formula:

  $$\cos 2\, x = 2\cos^2 x - 1 \Longrightarrow \cos^2 x = \frac{\cos 2\, x + 1}{2}. \tag{29}$$

  Thus

  $$\int \cos^2 x\, \mathrm{d}x = \int \left(\frac{\cos 2\, x}{2} + \frac{1}{2} \right) = \frac{\sin 2\, x}{4} + \frac{x}{2}. \tag{30}$$

Putting things together, the solution (of the new equation – obtained from the original through dividing $\cos^2 2\, y$) is given by

$$\frac{1}{2} \tan 2\, y = \frac{\sin 2\, x}{4} + \frac{x}{2} + C. \tag{31}$$

(You can choose to apply arctan to both sides, but that will make the formula look bad as when y is a solution, so is $y + \frac{k}{2}\, \pi$ for any integer k).

Finally we need to add back all the zeroes of $p(y) = \cos^2 2\, y$.

$$\cos^2 2\, y = 0 \Longleftrightarrow \cos 2\, y = 0 \Longleftrightarrow 2\, y = \left(k + \frac{1}{2} \right) \pi \Longleftrightarrow y = \frac{2\, k + 1}{4}\, \pi \tag{32}$$

for all integers k.[2]

Putting everything together, the solution to the original problem is

$$\frac{1}{2} \tan 2\, y = \frac{\sin 2\, x}{4} + \frac{x}{2} + C; \qquad y = \frac{2\, k + 1}{4}\, \pi \text{ for all integers } k. \tag{33}$$

Problem 4. (2.4 25) Let $y = y_1(t)$ be a solution of

$$y' + p(t)\, y = 0, \tag{34}$$

2. The book unnecessarily put $\pm$ before the ratio.

and let $y = y_2(t)$ be a solution of

$$y' + p(t)\, y = g(t). \tag{35}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of

$$y' + p(t)\, y = g(t). \tag{36}$$

Solution. y_1 is a solution of the homogeneous equation means

$$y_1' + p(t)\, y_1 = 0. \tag{37}$$

y_2 is a solution of the nonhomogeneous equation means

$$y_2' + p(t)\, y_2 = g(t). \tag{38}$$

Now we check

$$[y_1 + y_2]' + p(t)\,[y_1 + y_2] = y_1' + y_2' + p\, y_1 + p\, y_2 = \left(y_1' + p\, y_1\right) + \left(y_2' + p\, y_2\right) = 0 + g(t) = g(t). \tag{39}$$

So $y_1 + y_2$ is also a solution to the nonhomogeneous equation.

Problem 5. (2.6 3) Is the following equation exact? If it is, solve it.

$$\left(3\,x^2 - 2\,x\,y + 2\right) \mathrm{d}x + \left(6\,y^2 - x^2 + 3\right) \mathrm{d}y = 0. \tag{40}$$

Solution. This equation is already in the form $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$. Compute

$$\frac{\partial M}{\partial y} = -2\,x, \qquad \frac{\partial N}{\partial x} = -2\,x \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{41}$$

The equation is exact. We solve it by finding an $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M = 3\,x^2 - 2\,x\,y + 2, \qquad \frac{\partial u}{\partial y} = N = 6\,y^2 - x^2 + 3. \tag{42}$$

Using the first condition:

$$u(x, y) = \int \frac{\partial u}{\partial x}\,\mathrm{d}x + g(y) = x^3 - x^2\,y + 2\,x + g(y). \tag{43}$$

Then we use the second condition:

$$6\,y^2 - x^2 + 3 = N = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - x^2\,y + 2\,x + g(y)\right) = -x^2 + g'(y) \Longrightarrow g'(y) = 6\,y^2 + 3 \tag{44}$$

consequently

$$g(y) = 2\,y^3 + 3\,y. \tag{45}$$

So

$$u(x, y) = x^3 - x^2\,y + 2\,x + 2\,y^3 + 3\,y. \tag{46}$$

The general solution is given by

$$x^3 - x^2\,y + 2\,x + 2\,y^3 + 3\,y = C. \tag{47}$$

Problem 6. (2.6 15) Find the value b for which the equation is exact, and then solve it using that value of b.

$$\left(x\,y^2 + b\,x^2\,y\right) \mathrm{d}x + (x + y)\,x^2\,\mathrm{d}y = 0. \tag{48}$$

Solution. We compute

$$\frac{\partial M}{\partial y} = 2\,x\,y + b\,x^2; \qquad \frac{\partial N}{\partial x} = 3\,x^2 + 2\,x\,y. \tag{49}$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Longleftrightarrow b = 3. \tag{50}$$

The equation is exact if and only if $b = 3$.

For $b = 1$, we need u such that

$$\frac{\partial u}{\partial x} = x\,y^2 + 3\,x^2\,y; \qquad \frac{\partial u}{\partial y} = (x + y)\,x^2 = x^3 + x^2\,y. \tag{51}$$

Using the first:

$$u(x, y) = \frac{1}{2}\,x^2\,y^2 + x^3\,y + g(y). \tag{52}$$

Using the second:

$$x^3 + x^2\,y = \frac{\partial}{\partial y}\left(\frac{1}{2}\,x^2\,y^2 + x^3\,y + g(y)\right) = x^2\,y + x^3 + g'(y) \Longrightarrow g'(y) = 0 \tag{53}$$

So we can take $g(y) = 0$.

The final answer is

$$\frac{1}{2}\,x^2\,y^2 + x^3\,y = C. \tag{54}$$

INTERMEDIATE PROBLEMS

Problem 7. (2.6 25) Find an integrating factor and solve the equation.

$$(3\,x^2\,y+2\,x\,y+y^3)\,\mathrm{d}x+(x^2+y^2)\,\mathrm{d}y=0. \tag{55}$$

Solution. Compute

$$\frac{\partial M}{\partial y}=3\,x^2+2\,x+3\,y^2; \qquad \frac{\partial N}{\partial x}=2\,x \tag{56}$$

They are not equal, so the equation is not exact.

We need to find the integrating factor $\mu(x,y)$ such that

$$\frac{\partial}{\partial y}(\mu\,M)=\frac{\partial}{\partial x}(\mu\,N)\Longleftrightarrow M\,\frac{\partial\mu}{\partial y}-N\,\frac{\partial\mu}{\partial x}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu. \tag{57}$$

This is just

$$(3\,x^2\,y+2\,x\,y+y^3)\,\frac{\partial\mu}{\partial y}-(x^2+y^2)\,\frac{\partial\mu}{\partial x}=-\,3\,(x^2+y^2)\,\mu. \tag{58}$$

Let's guess.

- $\mu=\mu(x)$. This leads to

$$-\,(x^2+y^2)\,\mu'=-\,3\,(x^2+y^2)\,\mu\Longrightarrow\frac{\mu'}{\mu}=3 \tag{59}$$

  Therefore we can take

$$\mu=e^{3\,x}. \tag{60}$$

Multiply the equation by this μ we reach

$$\left[e^{3x}\,(3\,x^2\,y+2\,x\,y+y^3)\right]\mathrm{d}x+\left[e^{3x}\,(x^2+y^2)\right]\mathrm{d}y=0. \tag{61}$$

We can check

$$\frac{\partial}{\partial y}\left(e^{3x}\,(3\,x^2\,y+2\,x\,y+y^3)\right)=\frac{\partial}{\partial x}\left(e^{3x}\,(x^2+y^2)\right) \tag{62}$$

now.

Now we find u such that

$$\frac{\partial u}{\partial x}=e^{3x}\,(3\,x^2\,y+2\,x\,y+y^3); \qquad \frac{\partial u}{\partial y}=e^{3x}\,(x^2+y^2). \tag{63}$$

It is clear that performing $\int\ \frac{\partial u}{\partial y}\,\mathrm{d}y$ is much easier than doing $\int\ \frac{\partial u}{\partial x}\,\mathrm{d}x$. So we start from the second condition:

$$u=\int\ \frac{\partial u}{\partial y}\,\mathrm{d}y+g(x)=\int\ \left[e^{3x}\,x^2+e^{3x}\,y^2\right]\mathrm{d}y+g(x)=e^{3x}\,x^2\,y+\frac{1}{3}\,e^{3x}\,y^3+g(x). \tag{64}$$

Next using the first condition:

$$e^{3x}\,(3\,x^2\,y+2\,x\,y+y^3)=\frac{\partial u}{\partial x}=3\,e^{3x}\,x^2\,y+2\,e^{3x}\,x\,y+e^{3x}\,y^3+g'(x) \tag{65}$$

which leads to

$$g'(x)=0 \tag{66}$$

and we can take $g=0$.

Thus

$$u(x,y)=e^{3x}\,x^2\,y+\frac{1}{3}\,e^{3x}\,y^3 \tag{67}$$

and the solution to

$$\left[e^{3x}\,(3\,x^2\,y+2\,x\,y+y^3)\right]\mathrm{d}x+\left[e^{3x}\,(x^2+y^2)\right]\mathrm{d}y=0 \tag{68}$$

is given by

$$e^{3x}\,x^2\,y+\frac{1}{3}\,e^{3x}\,y^3=C. \tag{69}$$

As the multiplier $\mu(x,y)=e^{3x}$ does not contain y, there is no $y=y(x)$ such that $\mu(x,y(x))=0$ and therefore multiplying by μ does not change the solutions. So the solution to the original equation is also

$$e^{3x}\,x^2\,y+\frac{1}{3}\,e^{3x}\,y^3=C. \tag{70}$$

Problem 8. (2.6 27) Find an integrating factor and solve

$$\mathrm{d}x+(x/y-\sin y)\,\mathrm{d}y=0. \tag{71}$$

Solution. Compute

$$\frac{\partial M}{\partial y}=0, \qquad \frac{\partial N}{\partial x}=\frac{1}{y}. \tag{72}$$

We need μ such that

$$\frac{\partial}{\partial y}(\mu\,M)=\frac{\partial}{\partial x}(\mu\,N)\Longleftrightarrow M\,\frac{\partial\mu}{\partial y}-N\,\frac{\partial\mu}{\partial x}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu. \tag{73}$$

The equation for μ is then

$$\frac{\partial\mu}{\partial y} - (x/y - \sin y)\,\frac{\partial\mu}{\partial x} = \frac{1}{y}\,\mu. \tag{74}$$

This time it is clear that $\mu = \mu(y)$ would work:

$$\mu' = \frac{1}{y}\,\mu \tag{75}$$

and we can take

$$\mu = y. \tag{76}$$

Multiplying both sides of the equation by $\mu = y$ we have

$$y\,\mathrm{d}x + (x - y\sin y)\,\mathrm{d}y = 0. \tag{77}$$

We can check that it is exact now. We find u such that

$$\frac{\partial u}{\partial x} = y; \qquad \frac{\partial u}{\partial y} = x - y\sin y. \tag{78}$$

Clearly $\int \frac{\partial u}{\partial x}\,\mathrm{d}x$ is easier to do. So we use the first condition and write

$$u(x, y) = \int \frac{\partial u}{\partial x}\,\mathrm{d}x + g(y) = x\,y + g(y). \tag{79}$$

Now the second condition gives

$$x - y\sin y = \frac{\partial u}{\partial y} = x + g'(y) \Longrightarrow g'(y) = -\,y\sin y. \tag{80}$$

To find $g(y)$ we need integration by parts again:

$$\int f\,\mathrm{d}g = f\,g - \int g\,\mathrm{d}f. \tag{81}$$

This time we take $g = \sin y$.[3] [4]

$$\begin{aligned} g(y) &= -\int y\sin y\,\mathrm{d}y \\ &= \int y\,\mathrm{d}\cos y \\ &= y\cos y - \int \cos y\,\mathrm{d}y \\ &= y\cos y - \sin y. \end{aligned} \tag{83}$$

Therefore

$$u(x, y) = x\,y + y\cos y - \sin y. \tag{84}$$

The solution to the new equation (the one obtained by multiplying $\mu = y$) is

$$x\,y + y\cos y - \sin y = C. \tag{85}$$

Now we need to check those functions $y(x)$ such that $\mu(x, y(x)) = 0$. These are the solutions that are "brought in" by the multiplier and may not solve the original equation.[5] The only such function is the constant function $y = 0$. But the original equation involves x/y and thus $y = 0$ cannot be a solution.[6]

3. Another rule of thumb, whenever sin or cos is involved, put them behind d in $\int f\,\mathrm{d}g$.

4. What happens when both sin/cos and exp are there? Then either way is OK. One of the greatest discovery in Mathematics is that **sin/cos are just exp going complex.**

Example: Evaluate $\int e^t\sin t\,\mathrm{d}t$. The trick is to integrate by parts twice:

$$\begin{aligned} \int e^t\sin t\,\mathrm{d}t &= \int \sin t\,\mathrm{d}e^t \\ &= e^t\sin t - \int e^t\,\mathrm{d}\sin t \\ &= e^t\sin t - \int e^t\cos t\,\mathrm{d}t \\ &= e^t\sin t - \int \cos t\,\mathrm{d}e^t \\ &= e^t\sin t - e^t\cos t + \int e^t\,\mathrm{d}\cos t \\ &= e^t\sin t - e^t\cos t - \int e^t\sin t\,\mathrm{d}t. \end{aligned} \tag{82}$$

Now move the last term on the right hand side to the left... It is worth trying to start from $\int e^t\sin t\,\mathrm{d}t = -\int e^t\,\mathrm{d}\cos t$...

5. To understand how multiplying an equation can change solutions, consider the following simple examples. Consider $y' = x$. $y = 0$ is not a solution. But if we multiply both sides by y, the equation becomes $y\,y' = x\,y$ whose solutions are the same as that of the previous equation **except** that $y = 0$ "sneaks in"; On the other hand, if the equation we want to solve is $y\,y' = x\,y$, and we multiply both sides by $1/y$, then the solution $y = 0$ is lost.

So the final answer should be

$$x\,y + y\cos y - \sin y = C, \qquad \text{exclude } y = 0. \tag{86}$$

6. Even if one argues that $y = 0 \Longrightarrow \mathrm{d}y = 0$ and the x/y term disappears, we are still left with $\mathrm{d}x = 0$ which is not true.