Name: ID:

Math 317 Q1 Winter 2017 Quiz 6 Solutions

Apr. 7, 2017, 25 minutes

- The quiz has 3 problems. Total 10 + 1 points.

Question 1. (5 pts) *Let* $f(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. *Determine the domain of* f *and obtain an explicit formula for* f. *Justify your calculation.*

Solution. First we can calucate its radius of convergence:

$$R = \left(\limsup_{n \to \infty} \left| \frac{(-1)^n}{2n+1} \right|^{1/(2n+1)} \right)^{-1} = 1. \tag{1}$$

Therefore $f(x)$ is defined on $(-1, 1)$ but not defined on $[-1, 1]^c$. At the two end points we check

$$f(1) := \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \text{ converges, and} \tag{2}$$

$$f(-1) := \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} \text{ also converges.} \tag{3}$$

Therefore the domain of f is $[-1, 1]$.

To find the explicit formula, we calculate, for $x \in (-1, 1)$,

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}. \tag{4}$$

Consequently

$$f(x) = f(0) + \int_0^x \frac{\mathrm{d}t}{1+t^2} = f(0) + \arctan x. \tag{5}$$

As $f(0) = 0$ we conclude that $f(x) = \arctan x$ on $(-1, 1)$.

Finally, by Abel's theorem we know that $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ converges uniformly on $[-1, 1]$. As it is a power series, $f(x)$ is a continuous function on $[-1, 1]$. On the other hand, $\arctan x$ is also a continuous function on $[-1, 1]$. Consequently there must hold

$$f(x) = \arctan x, \qquad x \in [-1, 1]. \tag{6}$$

QUESTION 2. (5 PTS) *Calculate the Fourier series for the function* $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$.

Solution. We have

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \,\mathrm{d}x + \int_0^{\pi} x \,\mathrm{d}x\right] = -\frac{\pi}{2}, \tag{7}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos(n\,x)\,\mathrm{d}x + \int_0^{\pi} x\cos(n\,x)\,\mathrm{d}x\right] \\ &= \frac{1}{n^2\,\pi}\,[(-1)^n - 1]. \end{aligned} \tag{8}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin(n\,x)\,\mathrm{d}x + \int_0^{\pi} x\sin(n\,x)\,\mathrm{d}x\right] \\ &= \frac{1}{n}\,[1 - 2\,(-1)^n] \end{aligned}$$

$$f(x) \sim -\frac{\pi}{4} + \sum_{k=1}^{\infty}\left\{\frac{1}{n^2\,\pi}\,[(-1)^n - 1]\cos n\,x + \frac{1}{n}\,[1 - 2\,(-1)^n]\sin n\,x\right\} \tag{9}$$

QUESTION 3. (1 BONUS PT) *Find the sum* $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ *by considering the Fourier series of* e^{-x}.

Solution. We calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\,\mathrm{d}x = \frac{e^{\pi} - e^{-\pi}}{\pi}, \tag{10}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos n\,x\,\mathrm{d}x = \frac{(-1)^n}{n^2+1}\,\frac{e^{\pi} - e^{-\pi}}{\pi}, \tag{11}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\sin n\,x\,\mathrm{d}x = \frac{(-1)^n\,n}{n^2+1}\,\frac{e^{\pi} - e^{-\pi}}{\pi}. \tag{12}$$

Now by Parseval's identity we have

$$
\begin{aligned}
\frac{e^{2\pi}-e^{-2\pi}}{2}=\int_{-\pi}^{\pi}(e^{-x})^2\,\mathrm{d}x &= \pi\left[\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right] \\
&= \frac{1}{2}\frac{(e^{\pi}-e^{-\pi})^2}{\pi}+\sum_{n=1}^{\infty}\frac{1}{n^2+1}\frac{(e^{\pi}-e^{-\pi})^2}{\pi}. \qquad (13)
\end{aligned}
$$

Consequently

$$
\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{\pi}{2}\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}-\frac{1}{2}. \qquad (14)
$$